Shiroin package - function *prove*

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Jensen and weighted AM-GM inequalities

Jensen inequality

If $f : \mathbb{R}^k \to \mathbb{R}$ is a convex function, $v_1, \ldots, v_n \in \mathbb{R}^k$, $w_1, \ldots, w_n > 0$ and $w = \sum_{i=1}^n w_i$, then

$$\frac{\sum_{i=1}^n w_i f(v_i)}{w} \geq f\left(\frac{\sum_{i=1}^n v_i}{w}\right).$$

If $f(w_1, w_2, \ldots, w_n) = \prod_{i=1}^n x_i^{w_i}$ for some $x_1, \ldots, x_n > 0$, then we've got

$$\frac{\sum_{i=1}^n w_i x_i}{w} \geq \sqrt[w]{\prod_{i=1}^n x_i^{w_i}}.$$

This is called inequality of weighted arithmetic and geometric means (AM-GM inequality for short). We will use an equivalent inequality

$$\sum_{i=1}^n w_i x_i \geq w \prod_{i=1}^n x_i^{w_i/w}.$$

Example

Let $a, b > 0$. Prove that

$$30a^2b^2 + 60ab^4 \leq 48a^3 + 56b^6.$$

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We will use AM-GM inequality. Let $x_1 = a^3$, $x_2 = b^6$. We need to find such w_1, w_2, w_3, w_4, that

$$30a^2b^2 \leq w_1a^3 + w_2b^6$$

$$60ab^4 \leq w_3a^3 + w_4b^6$$

To fulfill assumptions of AM-GM, we need: $w_1, w_2, w_3, w_4 \geq 0$

$$w_1 + w_2 = 30$$

$$w_3 + w_4 = 60$$

$$(a^3)^{w_1/30}(b^6)^{w_2/30} = a^2b^2$$

$$(a^3)^{w_3/30}(b^6)^{w_4/30} = ab^4$$

We also need $w_1 + w_3 \leq 48$ and $w_2 + w_4 \leq 56$

Example

$$w_1, w_2, w_3, w_4 \geq 0$$

$$w_1 + w_2 = 30$$

$$w_3 + w_4 = 60$$

$$3w_1 = 30 \cdot 2$$

$$6w_2 = 30 \cdot 2$$

$$3w_3 = 30 \cdot 1$$

$$6w_4 = 30 \cdot 4$$

$$w_1 + w_3 \leq 48$$

$$w_2 + w_4 \leq 56$$

All these equations and inequalities are linear, so w_1, w_2, w_3, w_4 can be found using linear programming (in this case they can be found directly from equations, but this is not true in general case).

Problem

Prove that for all $x > 0$

$$4x^2 \leq 4x + x^3.$$

This inequality follows directly from AM-GM. But we can't use the same algorithm as in the previous problem.
Let's try to do this. We want to find $w_1, w_2 \geq 0$ such that

$$4x^2 \leq w_1 x + w_2 x^3$$

$$w_1 + w_2 = 4$$

$$1w_1 + 3w_2 = 4 \cdot 2$$

$$w_1 \leq 4$$

$$w_2 \leq 1$$

From the equations we can find that $w_1 = w_2 = 2$. But $w_2 \leq 1$, so there is no solution.

Substitutions

This problem can be solved by substituting x by $2y$. This way the problem can be reformulated.

Prove that for all $y > 0$

$$16y^2 \leq 8x + 8x^3.$$

This problem can be solved using the main algorithm.

Dealing with real coefficients in the proof

Prove that for all $x > 0$

$$4x^2 \leq 1 + 2x + 3x^3 + 4x^4.$$

This time the biggest problem is not finding a solution, but finding one with nice coefficients. For example, using weighted AM-GM we can show that

$$4x^2 \leq 2x + 2x^3$$

or

$$4x^2 \leq 1 + x + x^3 + x^4.$$

but we can also find a less nice solution like

$$4x^2 \leq 0.657238842 + 0.342761158x + 0.342761158x^3 + 0.657238842x^4.$$

Dealing with real coefficients in the proof

The simplest idea is to specify a goal function with random coefficients. Algorithm choose then (almost surely) an extreme point from the set of feasible solutions.

Set of solutions

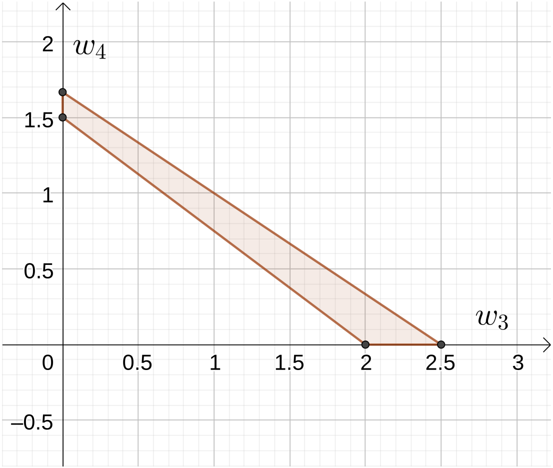

Figure: Set of solutions projected onto the plane $w_3 w_4$. w_1, w_2 are defined as $w_1 = 2w_3 + 3w_4 - 4$, $w_2 = 8 - 3w_3 - 4w_4$

Function *prove* - algorithm

1. Find a proof with real coefficients.
2. Check which inequalities have integer coefficients.
3. Subtract them from original inequality and go back to 1 with the new inequality.

The loop breaks when all coefficients are integer or when it has run fixed amount of times.