

examples

May 17, 2020

In this notebook there are presented examples of usage of shiroin, a python library for proving inequalities of multivariate polynomials.

At the beginning we need to load the packages.

```
In [1]: from sympy import *
        from shiroindev import *
        from IPython.display import Latex
        shiro.seed=1
        shiro.display=lambda x:display(Latex(x))
```

`shiro.seed=1` sets a seed for proving functions. If you don't write it, you can get a slightly different proof each time you run a function. This line is here only for the sake of reproducibility.

The next line provides a nicer display of proofs, i.e. formulas will be shown instead of LaTeX code of these formulas. Note that this works on Jupyter, but not on the git page.

Now let's make some proofs. We will use problems from <https://www.imomath.com/index.php?options=593&lmm=0>.

Problem 1 Prove the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$, if a, b, c are real numbers.

Function `prove` tries to prove that given formula is nonnegative, **assuming all variables are nonnegative**. In this case the nonnegativity assumption is not a problem, since all powers on the left side are even, so if $|a|^2 + |b|^2 + |c|^2 \geq |ab| + |ac| + |bc|$, then $a^2 + b^2 + c^2 = |a|^2 + |b|^2 + |c|^2 \geq |ab| + |ac| + |bc| \geq ab + ac + bc$.

```
In [2]: prove('(a^2+b^2+c^2-a*b-a*c-b*c)')
```

```
numerator: a2 - ab - ac + b2 - bc + c2
```

```
denominator: 1
```

```
status: 0
```

```
From weighted AM-GM inequality:
```

Program couldn't find a solution with integer coefficients. Try to multiple the formula by some integer and run this function again.

$$ab + ac + bc \leq a^2 + b^2 + c^2$$

```
Out [2]: 0
```

Function `prove` prints several things. The first two gives us a formula after expanding it. The next one is status, which is the return status of the first use of `scipy.optimize.linprog`. Possible outputs and explanations are

- 0 - found a proof with real coefficients,
- 1 - need more time,
- 2 - function didn't find a proof,
- 3,4 - loss of precision (which may happen if it has to work with big numbers).

Then we've got a hint. So let's use it!

In [3]: `prove('(a^2+b^2+c^2-a*b-a*c-b*c)*2')`

numerator: $2a^2 - 2ab - 2ac + 2b^2 - 2bc + 2c^2$

denominator: 1

status: 0

From weighted AM-GM inequality:

$$2ab \leq a^2 + b^2$$

$$2ac \leq a^2 + c^2$$

$$2bc \leq b^2 + c^2$$

$$0 \leq 0$$

The sum of all inequalities gives us a proof of the inequality.

Out [3]: 0

Problem 2 Find all real numbers such that $a^2 + b^2 + c^2 + d^2 = a(b + c + d)$.

At first glance it doesn't look like an inequality problem, but actually it is one. If you try to calculate both sides for different values, you can see that the left side of the equation is never less than the right one. So let's try

In [4]: `prove('a^2+b^2+c^2+d^2-a*(b+c+d)')`

numerator: $a^2 - ab - ac - ad + b^2 + c^2 + d^2$

denominator: 1

status: 2

Program couldn't find any proof.

$$ab + ac + ad \leq a^2 + b^2 + c^2 + d^2$$

Out [4]: 2

This time prove didn't found the proof. But it doesn't mean that the inequality is not true! prove uses a list of n positive values, where n is a number of variables in the formula. List of values should correspond to the list of variables in alphabetical order. Here are a few tips how to choose the right values.

1. Consider a function $pos(values)$ which is the sum of the positive addends in the formula after substitution of values to variables. Analogically, let $neg(values)$ be the sum of the negative addends. We should choose such values for which $quotient = pos(values)/|neg(values)|$ is small.
2. The symmetry group of the formula splits set of variables into orbits. Using the same values for variables in one orbit is recommended. In particular, if the symmetry group of the formula is transitive (for example, when the formula is cyclic), then all values (probably) should be the same.
3. If the formula is homogeneous, then $values = (a_1, a_2, \dots, a_n)$ provide a proof iff $values = (ka_1, ka_2, \dots, ka_n)$ provides a proof for any $k \in Q_+$ (as long as you don't run into overflow error).

In the formula above b, c, d are in one orbit and the formula is homogenous, so let's try $a = 2$ and $b = c = d = 1$.

In [5]: `prove('a^2+b^2+c^2+d^2-a*(b+c+d)', '2,1,1,1')`

Substitute $a \rightarrow 2e$

numerator: $b^2 - 2be + c^2 - 2ce + d^2 - 2de + 4e^2$

denominator: 1

status: 0

From weighted AM-GM inequality:

$$2be \leq b^2 + e^2$$

$$2ce \leq c^2 + e^2$$

$$2de \leq d^2 + e^2$$

$$0 \leq e^2$$

The sum of all inequalities gives us a proof of the inequality.

Out [5]: 0

Function makes a substitution $a \rightarrow 2e$ and try to prove new inequality. This time it succeeded. Moreover, if starting formula is equal to 0, then all these inequalities have to be equalities, so $e^2 = 0$ and eventually $a = 0$. We can also try a little bit lower value for a .

In [6]: `prove('a^2+b^2+c^2+d^2-a*(b+c+d)', '7/4,1,1,1')`

Substitute $a \rightarrow \frac{7f}{4}$

numerator: $16b^2 - 28bf + 16c^2 - 28cf + 16d^2 - 28df + 49f^2$

denominator: 16

status: 0

From weighted AM-GM inequality:

$$28bf \leq 14b^2 + 14f^2$$

$$28cf \leq 14c^2 + 14f^2$$

$$28df \leq 14d^2 + 14f^2$$

$$0 \leq 2b^2 + 2c^2 + 2d^2 + 7f^2$$

The sum of all inequalities gives us a proof of the inequality.

Out [6]: 0

Now we can see that if $a^2 + b^2 + c^2 + d^2 - a(b + c + d) = 0$, then $7f^2 + 2b^2 + 2c^2 + 2d^2 = 0$ and eventually $a = b = c = d = 0$. Note that inequality is proved only for positive numbers (which, by continuity, can be expanded to nonnegative numbers). But using similar argumentation to the one in previous problem, if $(a, b, c, d) = (x, y, z, t)$ is the solution of $a^2 + b^2 + c^2 + d^2 - a(b + c + d) = 0$, then $(a, b, c, d) = (|x|, |y|, |z|, |t|)$ is a solution, too. Since the only nonnegative solution is $(0, 0, 0, 0)$, it means that it is the only solution.

It is worth noting that this time function prove used f as a new variable instead of e . If you want to start a new proof and you don't care about the collision of variables from previous proofs, you can use `newproof` function, which clears the set of used variables.

We can also use the `findvalues` function to find values for the proof more automatically. It looks for (local) minimum of the *quotient* value defined above.

```
In [7]: formula=S('a^2+b^2+c^2+d^2-a*(b+c+d)')
        numvalues=findvalues(formula)
        numvalues
```

```
Optimization terminated successfully.
Current function value: 1.154701
Iterations: 68
Function evaluations: 127
```

```
Out [7]: (1.4339109663193974,
         0.8278441585048405,
         0.8279027492686561,
         0.8278930696996669)
```

If the *quotient* value were less than 1, that would mean that the formula is negative for given values. If *quotient* were equal to 1, then we have to choose exactly these values (or other values for which the *quotient* is equal to 1. But here *quotient* is greater than 1, so we can take a point near it and (probably) still have a proof. The values given to the prove function must not be floating point numbers, so we can rationalize them.

```
In [8]: values=nsimplify(numvalues,tolerance=0.1,rational=True)
        values
```

```
Out [8]:
      (10 5 5 5)
      ( 7 6 6 6)
```

In [9]: `newproof()`
`prove(formula, values)`

Substitute $a \rightarrow \frac{10e}{7}$

Substitute $b \rightarrow \frac{5f}{6}$

Substitute $c \rightarrow \frac{5g}{6}$

Substitute $d \rightarrow \frac{5h}{6}$

numerator: $3600e^2 - 2100ef - 2100eg - 2100eh + 1225f^2 + 1225g^2 + 1225h^2$

denominator: 1764

status: 0

From weighted AM-GM inequality:

$$2100ef \leq 1050e^2 + 1050f^2$$

$$2100eg \leq 1050e^2 + 1050g^2$$

$$2100eh \leq 1050e^2 + 1050h^2$$

$$0 \leq 450e^2 + 175f^2 + 175g^2 + 175h^2$$

The sum of all inequalities gives us a proof of the inequality.

Out [9]: 0

If you set the tolerance bigger, then the values will have smaller numerators and denominators, so coefficients in the proof will be smaller, too. But if the tolerance is too big, then proof will not be found.

Let's skip the problem 3 and look solve the problem 4 instead.

Problem 4 If x and y are two positive numbers less than 1, prove that

$$\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{1-xy}$$

In [10]: `prove('1/(1-x^2)+1/(1-y^2)-2/(1-x*y)')`

numerator: $-x^3y + 2x^2y^2 - x^2 - xy^3 + 2xy - y^2$

denominator: $x^3y^3 - x^3y - x^2y^2 + x^2 - xy^3 + xy + y^2 - 1$

status: 2

Program couldn't find any proof.

$$x^3y + x^2 + xy^3 + y^2 \leq 2x^2y^2 + 2xy$$

It looks like the formula is symmetric. You can assume without loss of generality that $x \geq y$.

Try

`prove(makesubs(S("-x**3*y + 2*x**2*y**2 - x**2 - x*y**3 + 2*x*y - y**2"),[(y, 'oo')]))`

Out [10]: 2

prove assumes that formula is well-defined if all variables are positive, so it doesn't have to analyze the denominator (except of choosing the right sign). In this case it is not true, since if $x = 1$, then $1 - x^2 = 0$. Also denominator is equal to $(x^2 - 1)(y^2 - 1)(xy - 1)$ which is negative for $x, y \in (0, 1)$. So we need to make some substitution after which new variables can have all positive values, not just these inside $(0, 1)$ interval.

We will use a function `makesubs` to generate these substitutions. It has three basic parameters: `formula`, `intervals` and `values`. `intervals` are current limitations of variables, `values` are values of variables for which *quotient* of formula is small. `values` should be inside corresponding intervals. This argument is optional but it's better to use it. Let's go back to our problem. If $x = y$, then $\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{1-xy}$, so it's the minimum value of the formula. So let `values=(1/2,1/2)` (**warning: do not use decimal point**, for example `'0.5,0.5'`).

```
In [11]: newproof()
         newformula,newvalues=makesubs('1/(1-x^2)+1/(1-y^2)-2/(1-x*y)', '[0,1],[0,1]', '1/2,1/2')
         prove(newformula*3,newvalues)
```

Substitute $x \rightarrow 1 - \frac{1}{a+1}$

Substitute $y \rightarrow 1 - \frac{1}{b+1}$

numerator: $6a^3b + 3a^3 - 12a^2b^2 - 3a^2b + 3a^2 + 6ab^3 - 3ab^2 - 6ab + 3b^3 + 3b^2$

denominator: $4a^2b + 2a^2 + 4ab^2 + 8ab + 3a + 2b^2 + 3b + 1$

status: 0

From weighted AM-GM inequality:

$$12a^2b^2 \leq 6a^3b + 6ab^3$$

$$3a^2b \leq 2a^3 + b^3$$

$$3ab^2 \leq a^3 + 2b^3$$

$$6ab \leq 3a^2 + 3b^2$$

$$0 \leq 0$$

The sum of all inequalities gives us a proof of the inequality.

```
Out [11]: 0
```

Now let's get back to problem 3.

Problem 3 If a, b, c are positive real numbers that satisfy $a^2 + b^2 + c^2 = 1$, find the minimal value of

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}$$

The problem is equivalent to finding minimum of $xy/z + yz/x + zx/y$ assuming $x + y + z = 1$ and $x, y, z > 0$. The first idea is to suppose that the minimum is reached when $x = y = z$. In that case, $x = y = z = 1/3$ and formula is equal to 1. Now we can substitute $z \rightarrow 1 - x - y$. Constraints for variables are $x > 0, y > 0, x + y < 1$. We can rewrite it as $x \in (0, 1 - y), y \in (0, 1)$.

These two conditions have two important properties: * constraints for variables are written as intervals, * there are no "backwards dependencies", i.e. there is no x in the interval of y .

If these two conditions hold, then you can use `makesubs` function.

```
In [12]: newproof()
         formula=Sm('xy/z+yz/x+zx/y-1').subs('z',S('1-x-y'))
         newformula,values=makesubs(formula,'[0,1-y],[0,1]','1/3,1/3')
         prove(newformula,values)
```

Substitute $x \rightarrow -y + 1 + \frac{y-1}{a+1}$

Substitute $y \rightarrow 1 - \frac{1}{b+1}$

Substitute $b \rightarrow \frac{c}{2}$

numerator: $a^4c^2 + a^3c^2 - 2a^3c - 4a^2c + 4a^2 + ac^2 - 2ac + c^2$

denominator: $a^3c^2 + 2a^3c + 2a^2c^2 + 4a^2c + ac^2 + 2ac$

status: 0

From weighted AM-GM inequality:

$$2a^3c \leq a^4c^2 + a^2$$

$$4a^2c \leq a^3c^2 + 2a^2 + ac^2$$

$$2ac \leq a^2 + c^2$$

$$0 \leq 0$$

The sum of all inequalities gives us a proof of the inequality.

Out [12]: 0

The proof is found, so the assumption that 1 is the minimum of $xy/z+yz/x+zx/y$ was good.

Functions `S` and `Sm` creates a SymPy object from a string. The only difference is that `Sm` assumes that there are no multi-letter variables and adds a multiplication sign between every two terms which has no operator sign, so object `Sm(xy/z+yz/x+zx/y)` has 3 variables x, y, z and `S('xy/z+yz/x+zx/y')` has 6 variables x, y, z, xy, yz, zx .

As you may have noticed, formulas are often cyclic or symmetric. Therefore you can use `cyclize` or `symmetrize` function to reduce the length of the written formula. Here are a few commands which will do the same as each other.

```
In [13]: prove('(a^2+b^2+c^2-a*b-a*c-b*c)*2')
         #prove(S('(a^2+b^2+c^2-a*b-a*c-b*c)*2'))
         #prove(Sm('2(a^2+b^2+c^2-ab-ac-bc)'))
         #prove(cyclize('2*a^2-2*a*b'))
         #prove(symmetrize('a^2-a*b'))
```

numerator: $2a^2 - 2ab - 2ac + 2b^2 - 2bc + 2c^2$

denominator: 1

status: 0

From weighted AM-GM inequality:

$$2ab \leq a^2 + b^2$$

$$2ac \leq a^2 + c^2$$

$$2bc \leq b^2 + c^2$$

$$0 \leq 0$$

The sum of all inequalities gives us a proof of the inequality.

Out [13]: 0

Now look at formula $(x - 1)^4$. It's quite obvious that it's nonnegative, but prove fails to show this!

In [14]: `prove('(x-1)^4')`

numerator: $x^4 - 4x^3 + 6x^2 - 4x + 1$

denominator: 1

status: 2

Program couldn't find any proof.

$$4x^3 + 4x \leq x^4 + 6x^2 + 1$$

Out [14]: 2

But there is a relatively simple method to generate a proof using this library. We will make two proofs: one for $x \in (1, \infty)$ and the second one for $(-\infty, 1)$.

In [15]: `newproof()
prove(makesubs('(x-1)^4', '(1,oo)'))`

Substitute $x \rightarrow a + 1$

numerator: a^4

denominator: 1

status: 0

$$0 \leq a^4$$

The sum of all inequalities gives us a proof of the inequality.

Out [15]: 0

In [16]: `newproof()
prove(makesubs('(x-1)^4', '(-oo,1)'))`

Substitute $x \rightarrow 1 - a$
 numerator: a^4
 denominator: 1
 status: 0

$$0 \leq a^4$$

The sum of all inequalities gives us a proof of the inequality.

Out [16]: 0

Now let's go to the problem 10 #### Problem 10 If $a, b, c, d > 0$, prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2.$$

Let's try a simple approach.

In [17]: `formula=cyclize('a/(b+c)',variables='a,b,c,d')-2`
`formula`

Out [17]:

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{a+d} + \frac{d}{a+b} - 2$$

In [18]: `prove(formula)`

numerator: $a^3c + a^3d + a^2b^2 - a^2bd - 2a^2c^2 - a^2cd + a^2d^2 + ab^3 - ab^2c - ab^2d - abc^2 + ac^3 - acd^2 + b^3d + b^2c^2 - 2b^2d^2 + bc^3 - bc^2d - bcd^2 + bd^3 + c^2d^2 + cd^3$

denominator: $a^2bc + a^2bd + a^2c^2 + a^2cd + ab^2c + ab^2d + abc^2 + 2abcd + abd^2 + ac^2d + acd^2 + b^2cd + b^2d^2 + bc^2d + bcd^2$

status: 2

Program couldn't find any proof.

$$a^2bd + 2a^2c^2 + a^2cd + ab^2c + ab^2d + abc^2 + acd^2 + 2b^2d^2 + bc^2d + bcd^2 \leq a^3c + a^3d + a^2b^2 + a^2d^2 + ab^3 + ac^3 + b^3d + b^3c + b^2c^2 + b^2d^2 + bc^3 + bcd^2 + cd^3$$

Out [18]: 2

This problem, like the previous one, can be solved by splitting the domain of variables to several subdomains. But we can also use the symmetry of this inequality. For example, without loss of generality we can assume that $a \geq c$ and $b \geq d$, so $a \in [c, \infty)$, $b \in [d, \infty)$.

In [19]: `newproof()`
`prove(makesubs(formula, '[c,oo],[d,oo]'))`

Substitute $a \rightarrow c + e$

Substitute $b \rightarrow d + f$

numerator: $c^2e^2 - c^2ef + c^2f^2 + 2cde^2 + 2cdf^2 + ce^3 + cef^2 + cf^3 + d^2e^2 + d^2ef + d^2f^2 + de^3 + de^2f + 2def^2 + df^3 + e^2f^2 + ef^3$

denominator: $c^4 + 4c^3d + 2c^3e + 2c^3f + 6c^2d^2 + 6c^2de + 6c^2df + c^2e^2 + 3c^2ef + c^2f^2 + 4cd^3 + 6cd^2e + 6cd^2f + 2cde^2 + 6cdef + 2cdf^2 + ce^2f + cef^2 + d^4 + 2d^3e + 2d^3f + d^2e^2 + 3d^2ef + d^2f^2 + de^2f + def^2$

status: 0

From weighted AM-GM inequality:

Program couldn't find a solution with integer coefficients. Try to multiple the formula by some integer and run this function again.

$$c^2ef \leq c^2e^2 + c^2f^2 + 2cde^2 + 2cdf^2 + ce^3 + cef^2 + cf^3 + d^2e^2 + d^2ef + d^2f^2 + de^3 + de^2f + 2def^2 + df^3 + e^2f^2 + ef^3$$

Out [19]: 0

In [20]: `newproof()`

`prove(makesubs(formula, '[c,oo],[d,oo]')*2)`

Substitute $a \rightarrow c + e$

Substitute $b \rightarrow d + f$

numerator: $2c^2e^2 - 2c^2ef + 2c^2f^2 + 4cde^2 + 4cdf^2 + 2ce^3 + 2cef^2 + 2cf^3 + 2d^2e^2 + 2d^2ef + 2d^2f^2 + 2de^3 + 2de^2f + 4def^2 + 2df^3 + 2e^2f^2 + 2ef^3$

denominator: $c^4 + 4c^3d + 2c^3e + 2c^3f + 6c^2d^2 + 6c^2de + 6c^2df + c^2e^2 + 3c^2ef + c^2f^2 + 4cd^3 + 6cd^2e + 6cd^2f + 2cde^2 + 6cdef + 2cdf^2 + ce^2f + cef^2 + d^4 + 2d^3e + 2d^3f + d^2e^2 + 3d^2ef + d^2f^2 + de^2f + def^2$

status: 0

From weighted AM-GM inequality:

$$2c^2ef \leq c^2e^2 + c^2f^2$$

$$0 \leq c^2e^2 + c^2f^2 + 4cde^2 + 4cdf^2 + 2ce^3 + 2cef^2 + 2cf^3 + 2d^2e^2 + 2d^2ef + 2d^2f^2 + 2de^3 + 2de^2f + 4def^2 + 2df^3 + 2e^2f^2 + 2ef^3$$

The sum of all inequalities gives us a proof of the inequality.

Out [20]: 0

It's a good idea to use intervals that are unbounded from one side (i.e. those which contain $\pm\infty$). In this problem we could assume that $a \in (0, c]$, $b \in (0, d]$ as well. But as you can see, in this case the proof is several times longer.

In [21]: `newproof()`

`prove(makesubs(formula, '[0,c],[0,d]')*2)`

Substitute $a \rightarrow c - \frac{c}{e+1}$

Substitute $b \rightarrow d - \frac{d}{f+1}$

numerator: $2c^4ef^3 + 6c^4ef^2 + 6c^4ef + 2c^4e - 2c^3de^2f^2 - 4c^3de^2f - 2c^3de^2 + 4c^3def^3 + 8c^3def^2 + 4c^3def + 2c^3df^3 + 4c^3df^2 + 2c^3df + 2c^2d^2e^3f + 2c^2d^2e^3 + 4c^2d^2e^2f + 4c^2d^2e^2 + 2c^2d^2ef^3 + 4c^2d^2ef^2 + 6c^2d^2ef + 4c^2d^2e + 2c^2d^2f^3 + 4c^2d^2f^2 + 4c^2d^2f + 2c^2d^2 + 4cd^3e^3f + 2cd^3e^3 + 2cd^3e^2f^2 + 12cd^3e^2f + 6cd^3e^2 + 4cd^3ef^2 + 12cd^3ef + 6cd^3e + 2cd^3f^2 + 4cd^3f + 2cd^3 + 2d^4e^3f + 6d^4e^2f + 6d^4ef + 2d^4f$