Contents

Lecture 1 Basic definitions February 25, 2019

Definition 1.1
A knot K in S^3 is a smooth (PL - smooth) embedding of a circle S^1 in S^3:

$$\varphi : S^1 \hookrightarrow S^3$$

Usually we think about a knot as an image of an embedding: $K = \varphi(S^1)$. Some basic examples and counterexamples are shown respectively in Figure 6 and Figure 2.

Figure 1: Knots examples: unknot (left) and trefoil (right).

Figure 2: Not-knots examples: an image of a function $S^1 \longrightarrow S^3$ that isn't injective (left) and of a function that isn't smooth (right).

Definition 1.2
Two knots $K_0 = \varphi_0(S^1)$, $K_1 = \varphi_1(S^1)$ are equivalent if the embeddings φ_0 and φ_1 are isotopic, that is there exists a continues function

$$\Phi : S^1 \times [0,1] \hookrightarrow S^3,$$
$$\Phi(x,t) = \Phi_t(x)$$

such that Φ_t is an embedding for any $t \in [0,1]$, $\Phi_0 = \varphi_0$ and $\Phi_1 = \varphi_1$.

Theorem 1.1
Two knots K_0 and K_1 are isotopic if and only if they are ambient isotopic,

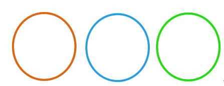
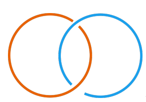
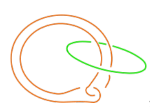
i.e. there exists a family of self-diffeomorphisms $\Psi = \{\psi_t : t \in [0,1]\}$ *such that:*

$$\begin{aligned}
&\psi(t) = \psi_t \text{ is continius on } t \in [0,1],\\
&\psi_t : S^3 \hookrightarrow S^3,\\
&\psi_0 = id,\\
&\psi_1(K_0) = K_1.
\end{aligned}$$

Definition 1.3
A knot is trivial (unknot) if it is equivalent to an embedding $\varphi(t) = (\cos t, \sin t, 0)$, *where* $t \in [0, 2\pi]$ *is a parametrisation of* S^1.

Definition 1.4
A link with k *- components is a (smooth) embedding of* $\overbrace{S^1 \sqcup ... \sqcup S^1}^{k}$ *in* S^3.

Example 1.1
Links:

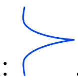
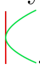
- *a trivial link with* 3 *components:* ,
- *a Hopf link:* ,
- *a Whitehead link:* ,
- *a Borromean link:* .

Definition 1.5
A link diagram D_π *is a picture over projection* π *of a link* L *in* $\mathbb{R}^3$ *(*S^3*) to* $\mathbb{R}^2$ *(*S^2*) such that:*

(1) $D_{\pi|_L}$ *is non degenerate,*

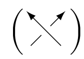
(2) the double points are not degenerate,

(3) there are no triple point.

By Definition 1.5 the following pictures can not be a part of a diagram: , , .

There are under- and overcrossings (tunnels and bridges) on a link diagrams with an obvious meaning.

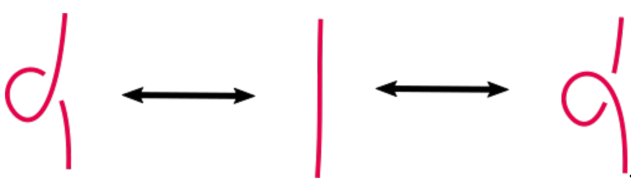
Lemma 1.1
Every link admits a link diagram.

Let D be a diagram of an oriented link (to each component of a link we add an arrow in the diagram). We can distinguish two types of crossings: right-handed , called a positive crossing, and left-handed , called a negative crossing.

Reidemeister moves

A Reidemeister move is one of the three types of operation on a link diagram as shown below:

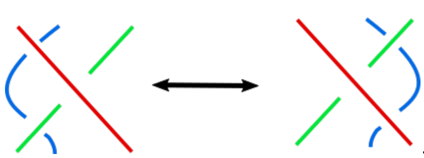
I

,

II

,

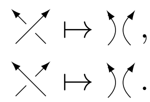
III

.

Theorem 1.2 (Reidemeister, 1927)
Two diagrams of the same link can be deformed into each other by a finite sequence of Reidemeister moves (and isotopy of the plane).

Seifert surface

Let D be an oriented diagram of a link L. We change the diagram by smoothing each crossing:



We smooth all the crossings, so we get a disjoint union of circles on the plane. Each circle bounds a disks in $\mathbb{R}^3$ (we choose disks that don't intersect). For each smoothed crossing we add a twisted band: right-handed for a positive and left-handed for a negative one. We get an orientable surface Σ such that $\partial\Sigma = L$.

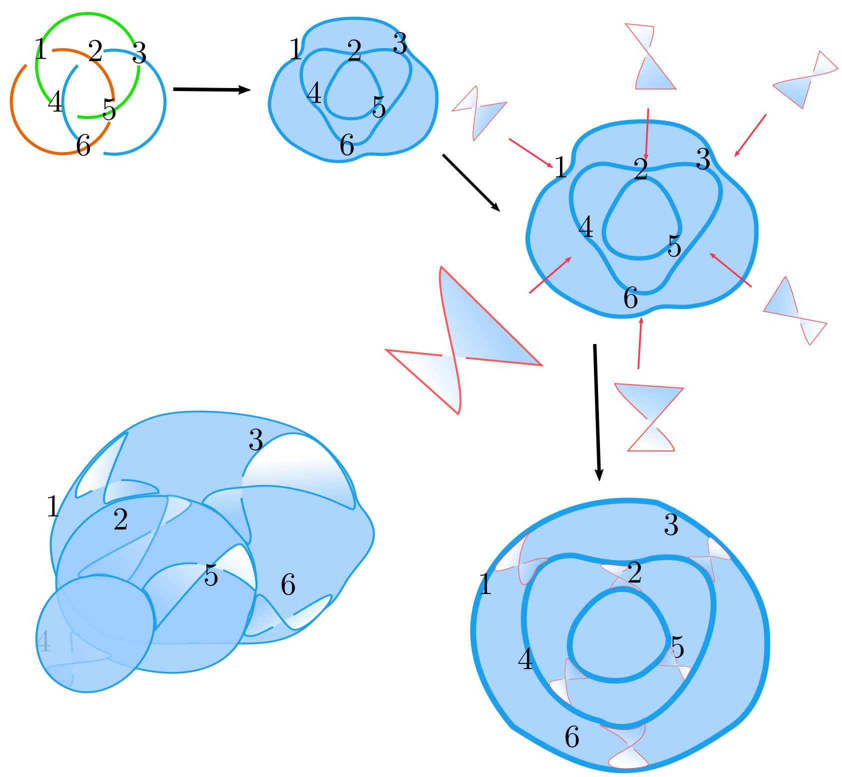


Figure 3: Constructing a Seifert surface.

Note: the obtained surface isn't unique and in general doesn't need to be connected, but by taking connected sum of all components we can easily get a connected surface (i.e. we take two disconnected components and cut a disk in each of them: D_1 and D_2. Then we glue both components on the boundaries: ∂D_1 and ∂D_2.

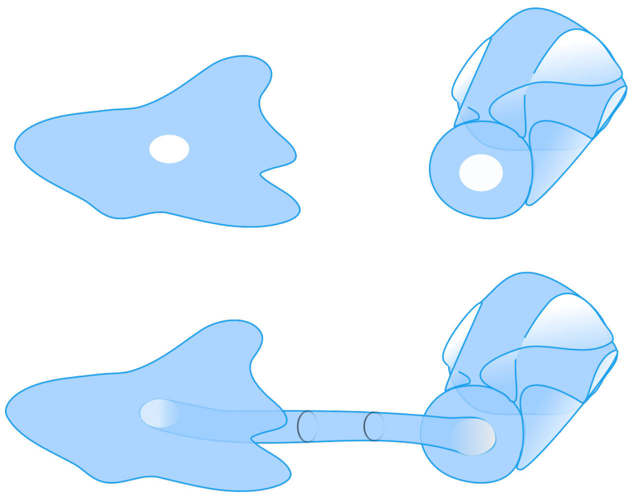


Figure 4: Connecting two surfaces.

Theorem 1.3 (Seifert)
Every link in S^3 bounds a surface Σ that is compact, connected and orientable. Such a surface is called a Seifert surface.

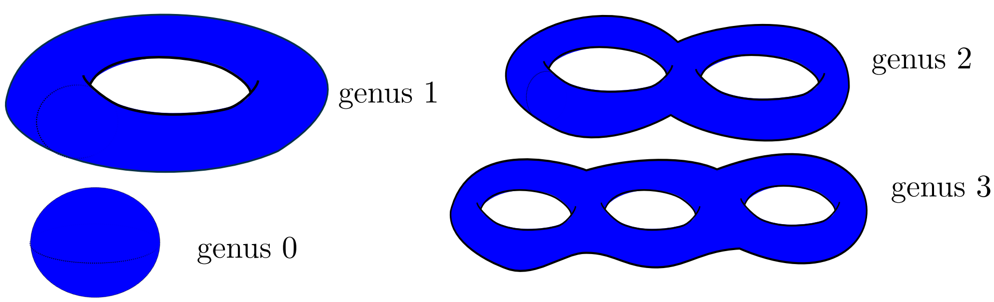


Figure 5: Genus of an orientable surface.

Definition 1.6
The three genus $g_3(K)$ ($g(K)$) of a knot K is the minimal genus of a Seifert surface Σ for K.

Corollary 1.1
A knot K is trivial if and only $g_3(K) = 0$.

Remark: there are knots that admit non isotopic Seifert surfaces of minimal genus (András Juhász, 2008).

Definition 1.7
Suppose α and β are two simple closed curves in $\mathbb{R}^3$. On a diagram L consider all crossings between α and β. Let N_+ be the number of positive crossings, N_- - negative. Then the linking number: $\mathrm{lk}(\alpha, \beta) = \frac{1}{2}(N_+ - N_-)$.

Definition 1.8
Let α and β be two disjoint simple closed curves in S^3. Let $\nu(\beta)$ be a tubular neighbourhood of β. The linking number can be interpreted via first homology group, where $\mathrm{lk}(\alpha, \beta)$ *is equal to evaluation of α as element of first homology group of the complement of β:*

$$\alpha \in H_1(S^3 \setminus \nu(\beta), \mathbb{Z}) \cong \mathbb{Z}.$$

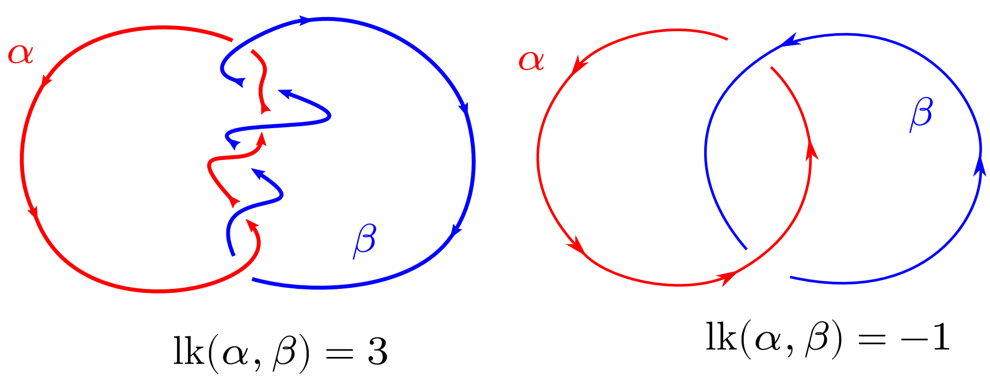


Figure 6: Linking number of a Hopf link (left) and a torus link $T(6, 2)$ (right).

Fact 1.1
$g_3(\Sigma) = \frac{1}{2} b_1(\Sigma) = \frac{1}{2} \dim_{\mathbb{R}} H_1(\Sigma, \mathbb{R})$, *where b_1 is first Betti number of a surface Σ.*

Seifert matrix

Let L be a link and Σ be an oriented Seifert surface for L. Choose a basis for $H_1(\Sigma, \mathbb{Z})$ consisting of simple closed curves $\alpha_1, \dots, \alpha_n$.
Let $\alpha_1^+, \dots \alpha_n^+$ be copies of α_i lifted up off the surface (push up along a vector field normal to Σ). Note that elements α_i are contained in the Seifert surface while all α_i^+ don't intersect the surface.
Let $\mathrm{lk}(\alpha_i, \alpha_j^+) = \{a_{ij}\}$. Then the matrix $S = \{a_{ij}\}_{i,j=1}^n$ is called a Seifert matrix for L. Note that by choosing a different basis we get a different matrix.

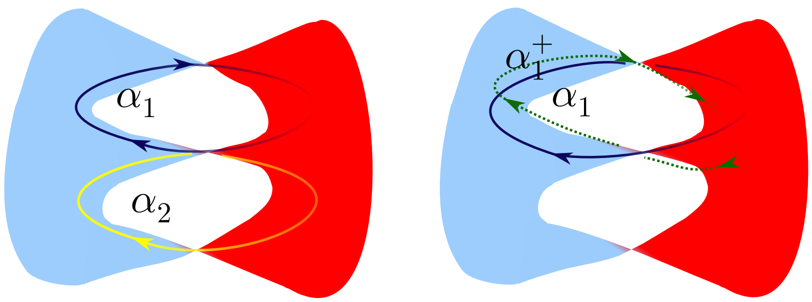


Figure 7: A basis α_1, α_2 of the first homology group of a Seifert surface and a copy of element α_1 pushed up along vector normal to the Seifert surface.

Theorem 1.4
The Seifert matrices S_1 and S_2 for the same link L are S-equivalent, that is, S_2 can be obtained from S_1 by a sequence of following moves:

(1) $V \to AVA^T$, *where A is a matrix with integer coefficients,*

(2) $$V \to \left(\begin{array}{ccc|cc} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right) \quad \textit{or} \quad V \to \left(\begin{array}{ccc|cc} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right),$$

(3) inverse of (2).

Existence of a Seifert surface - second proof

Proof. (Theorem 1.3)
Let $K \in S^3$ be a knot and $N = \nu(K)$ be its tubular neighbourhood. Because K and N are homotopy equivalent, we get:

$$H^1(S^3 \setminus N) \cong H^1(S^3 \setminus K).$$

Let us consider a long exact sequence of cohomology of a pair $(S^3, S^3 \setminus N)$ with integer coefficients:

$$\begin{array}{ccccc}
 & & \mathbb{Z} & & \\
 & & \wr\| & & \\
 & & H^0(S^3) \to & H^0(S^3 \setminus N) \to & \\
 & & & & \\
\to H^1(S^3, S^3 \setminus N) \to & & H^1(S^3) \to & H^1(S^3 \setminus N) \to & \\
 & & \wr\| & & \\
 & & 0 & & \\
 & & \wr\| & & \\
\to H^2(S^3, S^3 \setminus N) \to & & H^2(S^3) \to & H^2(S^3 \setminus N) \to & \\
 & & & & \\
\to H^3(S^3, S^3 \setminus N) \to & & H^3(S) \to & 0 & \\
 & & \wr\| & & \\
 & & \mathbb{Z} & &
\end{array}$$

The tubular neighbourhood of the knot is homomorphic to $D^2 \times S^1$. So its boundary $\partial N \cong S^1 \times S^1$ and therefore: $H^1(N, \partial N) \cong \mathbb{Z} \oplus \mathbb{Z}$. By excision theorem we have:

$$H^*(S^3, S^3 \setminus N) \cong H^*(N, \partial N).$$

Therefore:

$$H^1(S^3 \setminus N) \cong H^1(S^3 \setminus K) \cong \mathbb{Z}.$$

Let us consider the following diagram:

$$\begin{tikzcd}
H^1(S^3 \setminus K) \arrow[r] \arrow[d, "\widetilde{\Theta}"] & H^1(N \setminus K) \arrow[d, "\Theta"] \\
{[S^3 \setminus K, S^1]} \arrow[r] & {[N \setminus K, S^1]}
\end{tikzcd}$$

$\Sigma = \widetilde{\Theta}^{-1}(X)$ is a surface, such that $\partial\Sigma = K$, so it is a Seifert surface. □

Alexander polynomial

Definition 2.1
Let S be a Seifert matrix for a knot K. The Alexander polynomial $\Delta_K(t)$ is a Laurent polynomial:

$$\Delta_K(t) := \det(tS - S^T) \in \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$$

Theorem 2.1
$\Delta_K(t)$ is well defined up to multiplication by $\pm t^k$, for $k \in \mathbb{Z}$.

Proof. We need to show that $\Delta_K(t)$ doesn't depend on S-equivalence relation.

(1) Suppose $S' = CSC^T$, $C \in \mathrm{GL}(n, \mathbb{Z})$ (matrices invertible over $\mathbb{Z}$). Then $\det C = 1$ and:

$$\det(tS' - S'^T) = \det(tCSC^T - (CSC^T)^T) =$$
$$\det(tCSC^T - CS^TC^T) = \det C(tS - S^T)C^T = \det(tS - S^T)$$

(2) Let

$$A := t\left(\begin{array}{ccc|cc} & & & * & 0 \\ & S & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right) - \left(\begin{array}{ccc|cc} & & & * & 0 \\ & S^T & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} & & & * & 0 \\ & tS - S^T & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & -1 \\ 0 & \dots & 0 & t & 0 \end{array}\right)$$

Using the Laplace expansion we get $\det A = \pm t \det(tS - S^T)$.

□

Example 2.1
If K is a trefoil then we can take $S = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Then

$\Delta_K(t) = \det \begin{pmatrix} -t+1 & -t \\ 1 & -t+1 \end{pmatrix} = (t-1)^2+t = t^2-t+1 \neq 1 \Rightarrow$ *trefoil is not trivial.*

Fact 2.1
$\Delta_K(t)$ is symmetric.

Proof. Let S be an $n \times n$ matrix.

$$\Delta_K(t^{-1}) = \det(t^{-1}S - S^T) = (-t)^{-n} \det(tS^T - S) =$$
$$(-t)^{-n} \det(tS - S^T) = (-t)^{-n} \Delta_K(t)$$

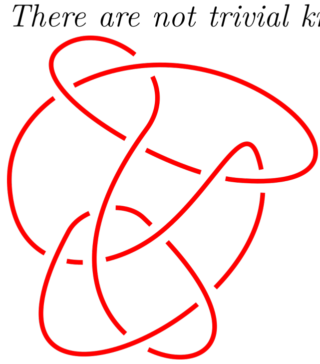
If K is a knot, then n is necessarily even, and so $\Delta_K(t^{-1}) = t^{-n}\Delta_K(t)$. $\square$

Lemma 2.1

$$\frac{1}{2} \deg \Delta_K(t) \leq g_3(K), \text{ where } deg(a_n t^n + \cdots + a_1 t^l) = k - l.$$

Proof. If Σ is a genus g - Seifert surface for K then $H_1(\Sigma) = \mathbb{Z}^{2g}$, so S is an $2g \times 2g$ matrix. Therefore $\det(tS - S^T)$ is a polynomial of degree at most $2g$. $\square$

Example 2.2
There are not trivial knots with Alexander polynomial equal 1, for example:



$\Delta_{11n34} \equiv 1.$

Decomposition of 3-sphere

We know that 3 - sphere can be obtained by gluing two solid tori:

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = (D^2 \times S^1) \cup (S^1 \times D^2).$$

So the complement of solid torus in S^3 is another solid torus.
Analytically it can be describes as follow.
Take $(z_1, z_2) \in \mathbb{C}$ such that $\max(|z_1|, |z_2|) = 1$. Define following sets:

$$S_1 = \{(z_1, z_2) \in S^3 : |z_1| = 0\} \cong S^1 \times D^2,$$
$$S_2 = \{(z_1, z_2) \in S^3 : |z_2| = 1\} \cong D^2 \times S^1.$$

The intersection $S_1 \cap S_2 = \{(z_1, z_2) : |z_1| = |z_2| = 1\} \cong S^1 \times S^1$.

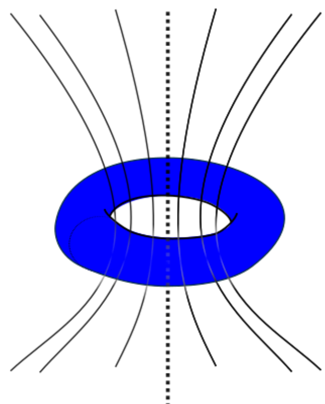


Figure 8: The complement of solid torus in S^3 is another solid torus.

Dehn lemma and sphere theorem

Lemma 2.2 (Dehn)
Let M be a 3-manifold and $D^2 \xrightarrow{f} M^3$ be a map of a disk such that $f\big|_{\partial D^2}$ is an embedding. Then there exists an embedding $D^2 \xhookrightarrow{g} M$ such that:

$$g\big|_{\partial D^2} = f\big|_{\partial D^2}.$$

Remark: Dehn lemma doesn't hold for dimension four.
Let M be connected, compact three manifold with boundary. Suppose $\pi_1(\partial M) \longrightarrow \pi_1(M)$ has non-trivial kernel. Then there exists a map $f : (D^2, \partial D^2) \longrightarrow (M, \partial M)$ such that $f\big|_{\partial D^2}$ is non-trivial loop in ∂M.

Theorem 2.2 (Sphere theorem)
Suppose $\pi_1(M) \neq 0$. Then there exists an embedding $f : S^2 \hookrightarrow M$ that is homotopy non-trivial.

Problem 2.1
Prove that S^3 K is Eilenberg–MacLane space of type $K(\pi, 1)$.

Corollary 2.1
Suppose $K \subset S^3$ and $\pi_1(S^3 \setminus K)$ is infinite cyclic ($\mathbb{Z}$). Then K is trivial.

Proof. Let N be a tubular neighbourhood of a knot K and $M = S^3 \setminus N$ its complement. Then $\partial M = S^1 \times S^1$. Let $f : \pi_1(\partial M) \longrightarrow \pi_1(M)$. If $\pi_1(M)$ is infinite cyclic group then the map f is non-trivial. Suppose $\lambda \in \ker(\pi_1(S^1 \times S^1) \longrightarrow \pi_1(M))$. There is a map $g : (D^2, \partial D^2) \longrightarrow (M, \partial M)$ such that $g(\partial D^2) = \lambda$.
By Dehn's lemma there exists an embedding $h : (D^2, \partial D^2) \hookrightarrow (M, \partial M)$ such that $h\big|_{\partial D^2} = f\big|_{\partial D^2}$ and $h(\partial D^2) = \lambda$. Let Σ be a union of the annulus and the image of ∂D^2. If $g_3(\Sigma) = 0$, then K is trivial.
Now we should proof that:

$$H_1(M) \cong \mathbb{Z} \Longrightarrow \lambda \in \ker(\pi_1(S^1 \times S^1) \longrightarrow \pi_1(M)).$$

Choose a meridian μ such that $\mathrm{lk}(\mu, K) = 1$. Recall the definition of linking

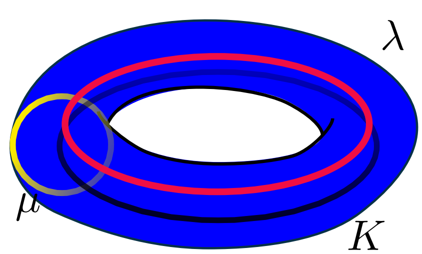


Figure 9: μ is a meridian and λ is a longitude.

number via homology group (Definition 1.8). $[\mu]$ represents the generator of $H_1(S^3 \setminus K, \mathbb{Z})$. From definition of λ we know that λ is trivial in $H_1(M)$ ($\mathrm{lk}(\lambda, K) = 0$, therefore $[\lambda]$ was trivial in $pi_1(M)$). If K is non-trivial then λ is non-trivial in $\pi_1(M)$, but it is trivial in $H_1(M)$. $\square$

Algebraic knots

Suppose $F : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial and $F(0) = 0$. Let take a small sphere S^3 around zero. This sphere intersect set of roots of F (zero set of F) transversally and by the implicit function theorem the intersection is a manifold. The dimension of sphere is 3 and $F^{-1}(0)$ has codimension 2. So there is a subspace L - compact one dimensional manifold without boundary. That means that L is a link in S^3.

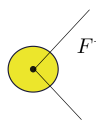


Figure 10: The intersection of a sphere S^3 and zero set of polynomial F is a link L.

Theorem 3.1
L is an unknot if and only if zero is a smooth point, i.e. $\bigtriangledown F(0) \neq 0$ (provided S^3 has a sufficiently small radius).

Remark: if S^3 is large it can happen that L is unlink, but $F^{-1}(0) \cap B^4$ is "complicated".
In other words: if we take sufficiently small sphere, the link is non-trivial if and only if the point 0 is singular and the isotopy type of the link doesn't depend on the radius of the sphere. A link obtained is such a way is called an algebraic link (in older books on knot theory there is another notion of algebraic link with another meaning).

Example 3.1
Let p and q be coprime numbers such that $p < q$ and $p, q > 1$.
Zero is an isolated singular point ($\bigtriangledown F(0) = 0$). F is quasi - homogeneous polynomial, so the isotopy class of the link doesn't depend on the choice of a sphere. Consider $S^3 = \{(z, w) \in \mathbb{C} : \max(|z|, |w|)\} = \varepsilon$. The intersection $F^{-1}(0) \cap S^3$ is a torus $T(p, q)$.
??????????????????? $F(z, w) = z^p - w^q$
.
$F^{-1}(0) = \{t = t^q, w = t^p\}$. *For unknot* $t = \max(|t|^p, |t|^q) = \varepsilon$.

as a corollary we see that $K_T^{n,}$????
is not slice unless $m = 0$.
$t = re^{i\Theta}, \Theta \in [0, 2\pi], r = \varepsilon^{\frac{i}{p}}$

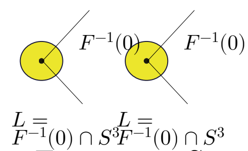


Figure 11: Sa.

Theorem 3.2
Suppose L is an algebraic link. $L = F^{-1}(0) \cap S^3$. Let

$$\varphi : S^3 \setminus L \longrightarrow S^1$$
$$\varphi(z, w) = \frac{F(z, w)}{|F(z, w)|} \in S^1, \quad (z, w) \notin F^{-1}(0).$$

The map φ is a locally trivial fibration.

???????
$rhD\varphi \equiv 1$

Definition 3.1
A map $\Pi : E \longrightarrow B$ is locally trivial fibration with fiber F if for any $b \in B$, there is a neighbourhood $U \subset B$ such that $\Pi^{-1}(U) \cong U\times$
?????????????
Γ *??????????????*
FIGURES
!!!!!!!!!!!!!!!!!!!!!!!!!!!

Theorem 3.3
The map $j : \mathcal{C} \longrightarrow \mathbb{Z}^{\infty}$ is a surjection that maps K_n to a linear independent set. Moreover $\mathcal{C} \cong \mathbb{Z}$

...
In general h is defined only up to homotopy, but this means that

$$h_* : H_1(F, \mathbb{Z}) \longrightarrow H_1(F, \mathbb{Z})$$

is well defined
???????????
map.

Theorem 3.4
Suppose S is a Seifert matrix associated with F then $h = S^{-1}S^T$.

Proof. TO WRITE REFERENCE!!!!!!!!!! □

Consequences:

(1) the Alexander polynomial is the characteristic polynomial of h:

$$\Delta_L(t) = \det(h - tId)$$

In particular Δ_L is monic (i.e. the top coefficient is ± 1), ???????????????

(2) S is invertible,

(3) F minimize the genus (i.e. F is minimal genus Seifert surface). ?????????????????

Definition 3.2
A link L is fibered if there exists a map $\phi : S^3 \setminus L \longrightarrow S^1$ which is locally trivial fibration.

If L is fibered then Theorem 3.4 holds and all its consequences.

Problem 3.1
If K_1 and K_2 are fibered knots, then also $K_1 \# K_2$ is fibered.

????????????????????

Problem 3.2
Prove that connected sum is well defined:
$\Delta_{K_1 \# K_2} = \Delta_{K_1} + \Delta_{K_2}$ *and* $g_3(K_1 \# K_2) = g_3(K_1) + g_3(K_2)$.

Alternating knot

Definition 3.3
A knot (link) is called alternating if it admits an alternating diagram.

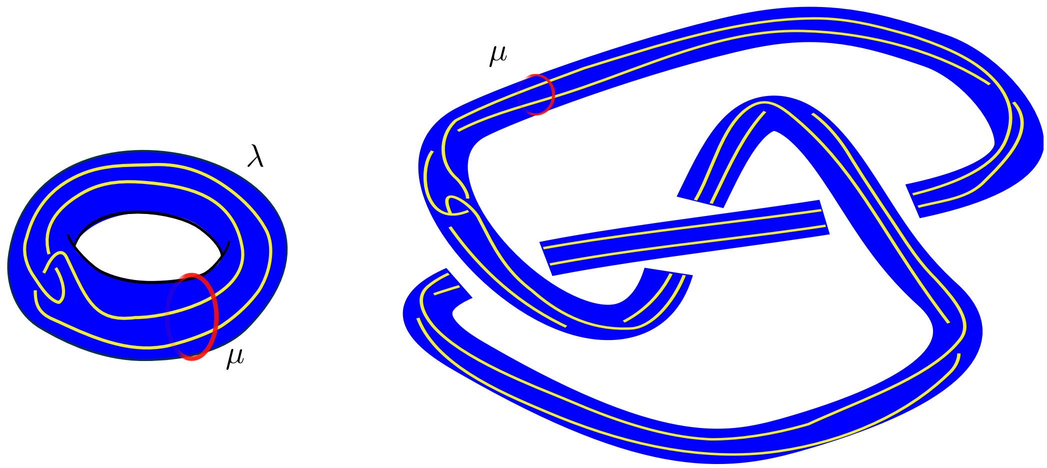


Figure 12: Example for a satellite knot: a Whitehead double of a trefoil.
The pattern knot embedded non-trivially in an unknotted solid torus T (e.i. $K \not\subset S^3 \subset T$) on the left and the pattern in a companion knot - trefoil - on the right.

Definition 3.4
A reducible crossing in a knot diagram is a crossing for which we can find a circle such that its intersection with a knot diagram is exactly that crossing. A knot diagram without reducible crossing is called reduced.

Fact 3.1
Any reduced alternating diagram has minimal number of crossings.

Definition 3.5
The writhe of the diagram is the difference between the number of positive and negative crossings.

Fact 3.2 (Tait)
Any two diagrams of the same alternating knot have the same writhe.

Fact 3.3
An alternating knot has Alexander polynomial of the form: $a_1 t^{n_1} + a_2 t^{n_2} + \cdots + a_s t^{n_s}$, *where* $n_1 < n_2 < \cdots < n_s$ *and* $a_i a_{i+1} < 0$.

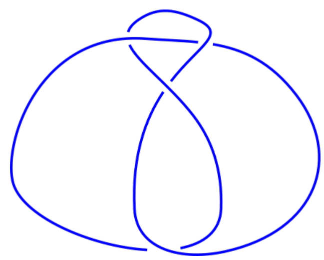


Figure 13: Example: figure eight knot is an alternating knot.

Problem 3.3 (open)
What is the minimal $\alpha \in \mathbb{R}$ *such that if* z *is a root of the Alexander polynomial of an alternating knot, then* $\Re(z) > \alpha$.
Remark: alternating knots have very simple knot homologies.

Proposition 3.1
If $T_{p,q}$ *is a torus knot,* $p < q$*, then it is alternating if and only if* $p = 2$.

Lecture 4 Concordance group March 18, 2019

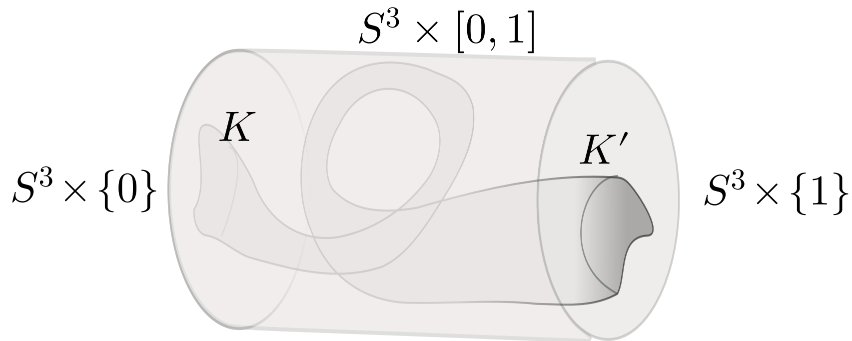
Definition 4.1
Two knots K *and* K' *are called (smoothly) concordant if there exists an annulus* A *that is smoothly embedded in* $S^3 \times [0,1]$ *such that*

$$\partial A = K' \times \{1\} \sqcup K \times \{0\}.$$

Definition 4.2
A knot K *is called (smoothly) slice if* K *is smoothly concordant to an unknot. Put differently: a knot* K *is smoothly slice if and only if* K *bounds a smoothly embedded disk in* B^4.

Let $m(K)$ denote a mirror image of a knot K.

Fact 4.1
For any K*,* $K\#m(K)$ *is slice.*



Fact 4.2
Concordance is an equivalence relation.

Fact 4.3
If $K_1 \sim K_1{}'$ and $K_2 \sim K_2{}'$, then $K_1 \# K_2 \sim K_1{}' \# K_2{}'$.

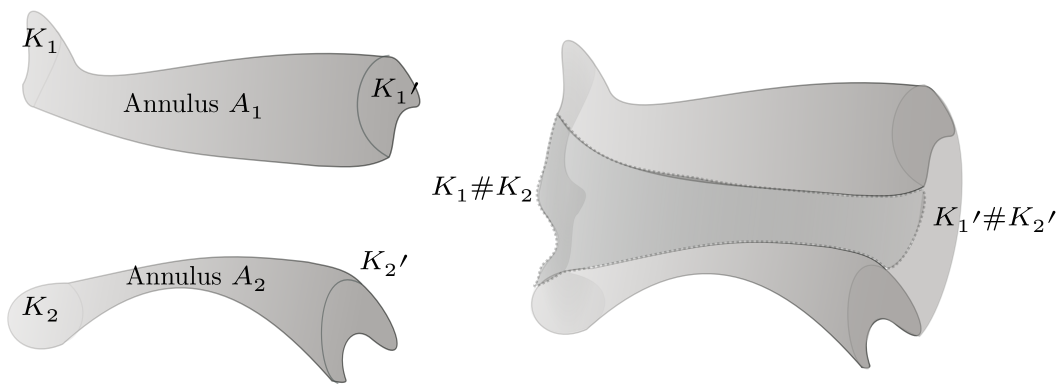


Figure 14: Sketch for Fact 4.3.

Fact 4.4
$K \# m(K) \sim$ the unknot.

Theorem 4.1
Let $\mathcal{C}$ denote a set of all equivalent classes for knots and $[0]$ denote class of all knots concordant to a trivial knot. $\mathcal{C}$ is a group under taking connected sums. The neutral element in the group is $[0]$ and the inverse element of an element $[K] \in \mathcal{C}$ is $-[K] = [mK]$.

Fact 4.5
The figure eight knot is a torsion element in $\mathcal{C}$ ($2K \sim$ the unknot).

Problem 4.1 (open)
Are there in concordance group torsion elements that are not 2 torsion elements?

Remark: $K \sim K' \Leftrightarrow K\# -K'$ is slice.

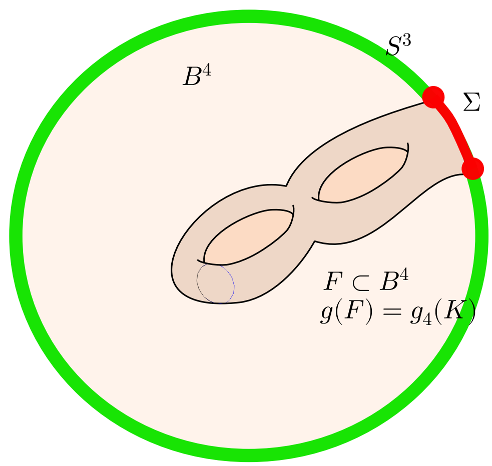


Figure 15: $Y = F \cup \Sigma$ is a smooth closed surface.

Pontryagin-Thom construction tells us that there exists a compact oriented three - manifold $\Omega \subset B^4$ such that $\partial\Omega = Y$.
Suppose Σ is a Seifert surface and V a Seifert form defined on Σ: $(\alpha, \beta) \mapsto \mathrm{lk}(\alpha, \beta^+)$.
Suppose $\alpha, \beta \in H_1(\Sigma, \mathbb{Z})$, i.e. there are cycles and

$$\alpha, \beta \in \ker(H_1(\Sigma, \mathbb{Z}) \longrightarrow H_1(\Omega, \mathbb{Z})).$$

Then there are two cycles $A, B \in \Omega$ such that $\partial A = \alpha$ and $\partial B = \beta$. Let B^+ be a push off of B in the positive normal direction such that $\partial B^+ = \beta^+$. Then $\mathrm{lk}(\alpha, \beta^+) = A \cdot B^+$. But A and B are disjoint, so $\mathrm{lk}(\alpha, \beta^+) = 0$. Then the Seifert form is zero.
Let us consider following maps:

$$\Sigma \overset{\phi}{\hookrightarrow} Y \overset{\psi}{\hookrightarrow} \Omega.$$

Let ϕ_* and ψ_* be induced maps on the homology group. If an element $\gamma \in \ker(H_1(\Sigma, \mathbb{Z}) \longrightarrow H_1(\Omega, \mathbb{Z}))$, then $\gamma \in \ker \phi_*$ or $\gamma \in \ker \psi_*$.

Proposition 4.1

$$\dim \ker(H_1(Y,\mathbb{Z}) \longrightarrow H_1(\Omega,\mathbb{Z})) = \frac{1}{2} b_1(Y),$$

where b_1 is first Betti number.

Proof. Consider the following long exact sequence for a pair (Ω, Y):

$$\begin{aligned} 0 \to H_3(\Omega) \to H_3(\Omega, Y) \to \\ \to H_2(Y) \to H_2(\Omega) \to H_2(\Omega, Y) \to \\ \to H_1(Y) \to H_1(\Omega) \to H_1(\Omega, Y) \to \\ \to H_0(Y) \to H_0(\Omega) \to 0 \end{aligned}$$

By Poincaré duality we know that:

$$\begin{aligned} H_3(\Omega, Y) &\cong H^0(\Omega), \\ H_2(Y) &\cong H^0(Y), \\ H_2(\Omega) &\cong H^1(\Omega, Y), \\ H_1(\Omega, Y) &\cong H^1(\Omega). \end{aligned}$$

Therefore $\dim_{\mathbb{Q}} {H_1(Y)}/{V} = \dim_{\mathbb{Q}} V$.
Suppose $g(K) = 0$ (K is slice). Then $H_1(\Sigma, \mathbb{Z}) \cong H_1(Y, \mathbb{Z})$. Let g_Σ be the genus of Σ, $\dim H_1(Y,\mathbb{Z}) = 2g_\Sigma$. Then the Seifert form V on a K has a subspace of dimension g_Σ on which it is zero:

$$V = \begin{array}{cc} & \overbrace{\hphantom{0 \;\; \dots \;\; 0}}^{g_\Sigma} \\ g_\Sigma \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \\ & \end{array} \begin{pmatrix} 0 & \dots & 0 & * & \dots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & * & \dots & * \\ * & \dots & * & * & \dots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \dots & * & * & \dots & * \end{pmatrix}_{2g_\Sigma \times 2g_\Sigma}$$

$\square$

Let $V = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$. Then

$$tV - V^T = \begin{pmatrix} 0 & tA \\ tB & tC \end{pmatrix} - \begin{pmatrix} 0 & B^T \\ A^T & C^T \end{pmatrix} = \begin{pmatrix} 0 & tA - B^T \\ tB - A^T & tC - C^T \end{pmatrix}$$
$$\det(tV - V^T) = \det(tA - B^T) - \det(tB - A^T)$$

Corollary 4.1
If K is a slice knot then there exists $f \in \mathbb{Z}[t, t^{-1}]$ such that

$$\Delta_K(t) = f(t) \cdot f(t^{-1}).$$

Example 4.1
Figure eight knot is not slice.

Fact 4.6
If K is slice, then the signature $\sigma(K) \equiv 0$:

$$V + V^T = \begin{pmatrix} 0 & A + B^T \\ B + A^T & C + C^T \end{pmatrix} \Rightarrow \sigma = 0.$$

Lecture 5 Genus g cobordism — March 25, 2019

Slice knots and metabolic form

Theorem 5.1
If K is slice, then $\sigma_K(t) = \operatorname{sign}((1-t)S + (1-\bar{t})S^T)$ is zero except possibly of finitely many points and $\sigma_K(-1) = \operatorname{sign}(S + S^T) \neq 0$.

Lemma 5.1
If V is a Hermitian matrix ($\overline{V} = V^T$) of size $2n \times 2n$, $V = \begin{pmatrix} 0 & A \\ \overline{A^T} & B \end{pmatrix}$ and $\det V \neq 0$ then $\sigma(V) = 0$.

Definition 5.1
A Hermitian form V is metabolic if V has structure $\begin{pmatrix} 0 & A \\ A^T & B \end{pmatrix}$ with half-dimensional null-space.

Theorem 5.1 can be also express as follow: non-degenerate metabolic hermitian form has vanishing signature.

Proof. We note that $\det(S + S^T) \neq 0$. Hence $\det((1-t)S + (1-\bar{t})S^T)$ is not identically zero on S^1, so it is non-zero except possibly at finitely many points. We apply the Lemma 5.1.
Let $t \in S^1 \setminus \{1\}$. Then:

$$\begin{aligned}\det((1-t)S + (1-\bar{t})S^T) = & \det((1-t)S + (t\bar{t} - \bar{t})S^T) = \\ & \det((1-t)(S - \bar{t} - S^T)) = \det((1-t)(S - \bar{t}S^T)).\end{aligned}$$

As $\det(S + S^T) \neq 0$, so $S - \bar{t}S^T \neq 0$. □

Corollary 5.1
If $K \sim K'$ then for all but finitely many $t \in S^1 \setminus \{1\} : \sigma_K(t) = -\sigma_{K'}(t)$.

Proof. If $K \sim K'$ then $K\#K'$ is slice.

$$\sigma_{-K'}(t) = -\sigma_{K'}(t)$$

The signature gives a homomorphism from the concordance group to $\mathbb{Z}$. Remark: if $t \in S^1$ is not algebraic over $\mathbb{Z}$, then $\sigma_K(t) \neq 0$ (we can use the argument that $\mathcal{C} \longrightarrow \mathbb{Z}$ as well). □

Four genus

Proposition 5.1 (Kawauchi inequality)
If there exists a genus g surface as in Figure 16 then for almost all $t \in S^1 \setminus \{1\}$ we have $|\sigma_K(t) - \sigma_{K'}(t)| \leq 2g$.

Lemma 5.2
If K bounds a genus g surface $X \in B^4$ and S is a Seifert form then $S \in M_{2n \times 2n}$ has a block structure $\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$, where 0 is $(n-g) \times (n-g)$ submatrix.

Figure 16: K and K' are connected by a genus g surface.

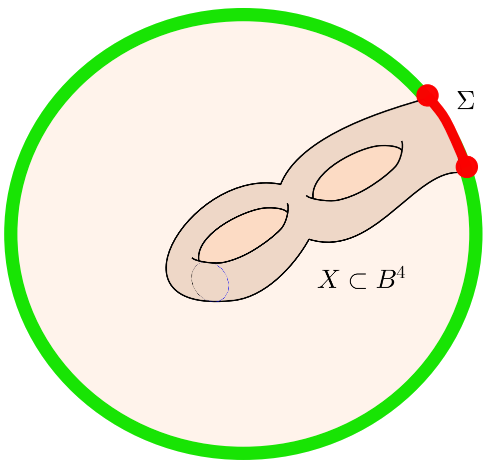


Figure 17: There exists a 3 - manifold Ω such that $\partial\Omega = X \cup \Sigma$.

Proof. Let K be a knot and Σ its Seifert surface as in Figure 17. There exists a 3 - submanifold Ω such that $\partial\Omega = Y = X \cup \Sigma$ (by Thom-Pontryagin construction). If $\alpha, \beta \in \ker(H_1(\Sigma) \longrightarrow H_1(\Omega))$, then $\mathrm{lk}(\alpha, \beta^+) = 0$. Now we have to determine the size of the kernel. We know that $\dim H_1(\Sigma) = 2n$. When we glue Σ (genus n) and X (genus g) along a circle we get a surface of genus $n + g$. Therefore $\dim H_1(Y) = 2n + 2g$. Then:

$$\dim(\ker(H_1(Y) \longrightarrow H_1(\Omega)) = n + g.$$

So we have $H_1(W)$ of dimension $2n + 2g$ - the image of $H_1(Y)$ with a subspace corresponding to the image of $H_1(\Sigma)$ with dimension $2n$ and a subspace corresponding to the kernel of $H_1(Y) \longrightarrow H_1(\Omega)$ of size $n + g$. We consider minimal possible intersection of this subspaces that corresponds to the kernel of the composition $H_1(\Sigma) \longrightarrow H_1(Y) \longrightarrow H_1(\Omega)$. As the first map is injective, elements of the kernel of the composition have to be in the kernel of the second map. So we can calculate:

$$\dim \ker(H_1(\Sigma) \longrightarrow H_1(\Omega)) = 2n + n + g - 2n - 2g = n - g.$$

$\square$

Corollary 5.2
If t is not a root of $\det(tS - S^T)$, then $|\sigma_K(t)| \leq 2g$.

Fact 5.1
If there exists cobordism of genus g between K and K' like shown in Figure 18, then $K\# - K'$ bounds a surface of genus g in B^4.

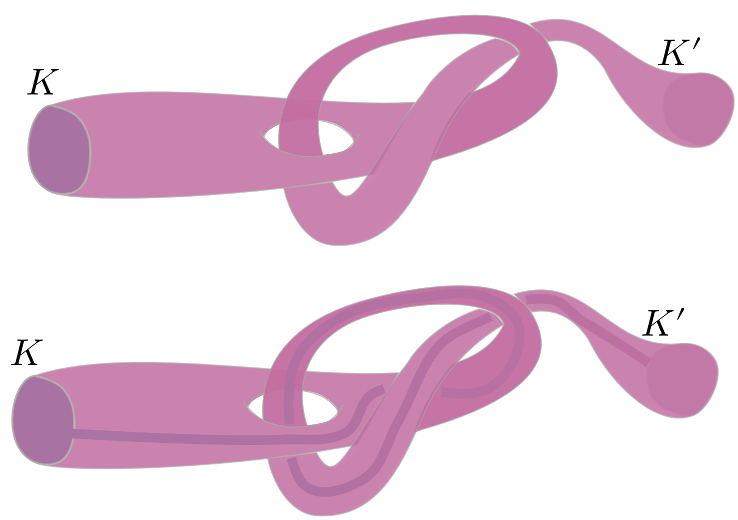


Figure 18: If K and K' are connected by a genus g surface, then $K\# - K'$ bounds a genus g surface.

Definition 5.2
The (smooth) four genus $g_4(K)$ is the minimal genus of the surface $\Sigma \in B^4$ such that Σ is compact, orientable and $\partial\Sigma = K$.

Remarks:

(1) 3 - genus is additive under taking connected sum, but 4 - genus is not,

(2) for any knot K we have $g_4(K) \leq g_3(K)$.

Example 5.1

- *Let* $K = T(2,3)$. $\sigma(K) = -2$, *therefore* $T(2,3)$ *isn't a slice knot.*
- *Let* K *be a trefoil and* K' *a mirror of a trefoil.* $g_4(K') = 1$, *but* $g_4(K\#K') = 0$, *so we see that* 4*-genus isn't additive,*
- *the equality:*

$$g_4(T(p,q)) = \frac{1}{2}(p-1)(g-1)$$

was conjecture in the '70 and proved by P. Kronheimer and T. Mrówka (1994).

Proposition 5.2
$g_4(T(p,q)\# - T(r,s))$ *is in general hopelessly unknown.*

Proposition 5.3
Supremum of the signature function of the knot is bounded almost everywhere by two times 4 *- genus:*

$$\operatorname{ess\,sup} |\sigma_K(t)| \leq 2g_4(K).$$

Topological genus

Definition 5.3
A knot K *is called topologically slice if* K *bounds a topological locally flat disc in* B^4 *(i.e. the disk has tubular neighbourhood).*

Theorem 5.2 (Freedman, '82)
If $\Delta_K(t) = 1$, *then* K *is topologically slice (but not necessarily smoothly slice).*

Theorem 5.3 (Powell, 2015)
If K *is genus* g *(topologically flat) cobordant to* K', *then*

$$|\sigma_K(t) - \sigma_{K'}(t)| \leq 2g$$

if $g_4^{\text{top}}(K) \geq \operatorname{ess\,sup} |\sigma_K(t)|$.

The proof for smooth category was based on following equality:

$$\dim \ker(H_1(Y) \longrightarrow H_1(\Omega)) = \frac{1}{2} \dim H_1(Y).$$

For this equality we assumed that there exists a 3 - dimensional manifold Ω (as shown in Figure 17) which was guaranteed by Pontryagin-Thom Construction.
Pontryagin-Thom Construction relays on taking Ω as preimage of regular value:

$$H^1(B^4 \setminus Y, \mathbb{Z}) = [B^4 \setminus Y, S^1],$$

what relies on Sard's theorem, that the set of regular values has positive measure. But Sard's theorem doesn't work for topologically locally flat category. So there was a gap in the proof for topological locally flat category - the existence of Ω.
Remark: unless $p = 2$ or $p = 3 \wedge q = 4$:

$$g_4^{\text{top}}(T(p,q)) < q_4(T(p,q)).$$

From the category of cobordant knots (or topologically cobordant knots) there exists a map to $\mathbb{Z}$ given by signature function. To any element K we can associate a form

$$(1-t)S + (1-\bar{t})S^T) \in W(\mathbb{Z}[t, t^{-1}]).$$

This association is not well define because id depends on the choice of Seifert form. However, different choices lead ever to congruent forms ($S \mapsto CSC^T$) or induced the change on the form by adding or subtracting a hyperbolic element.

Definition 5.4
The Witt group W of $\mathbb{Z}[t, t^{-1}]$ elements are classes of non-degenerate forms over $\mathbb{Z}[t, t^{-1}]$ under the equivalence relation $V \sim W$ if $V \oplus -W$ is metabolic.

If S differs from S' by a row extension, then $(1-t)S + (1-\bar{t}^{-1})S^T$ is Witt equivalence to $(1-t)S' + (1-t^{-1})S^T$.
A form is meant as hermitian with respect to this involution: $\bar{A}^T = A$: $(a, b) = (\bar{a}, b)$.
$W(\mathbb{Z}_p) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$

??????????????????????
$\sum a_g t^j \longrightarrow \sum a_g t^{-1}$

Theorem 5.4 (Levine '68)

$$W(\mathbb{Z}[t^{\pm 1}]) \longrightarrow \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}$$

Lecture 6 April 8, 2019

X is a closed orientable four-manifold. For simplicity assume $\pi_1(X) = 0$ (it is not needed to define the intersection form). In particular $H_1(X) = 0$. H_2 is free (exercise).

$$H_2(X, \mathbb{Z}) \xrightarrow{\text{Poincaré duality}} H^2(X, \mathbb{Z}) \xrightarrow{\text{evaluation}} \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}).$$

Intersection form: $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ is symmetric and non singular. Let A and B be closed, oriented surfaces in X.
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$$\begin{aligned} x &\in A \cap B \\ T_X A \oplus T_X B &= T_X X \\ \{\epsilon_1, \dots, \epsilon_n\} &= A \cap C \\ A \cdot B &= \sum_{i=1}^{n} \epsilon_i \end{aligned}$$

Proposition 6.1
Intersection form $A \cdot B$ doesn't depend of choice of A and B in their homology classes:

$$[A], [B] \in H_2(X, \mathbb{Z}).$$

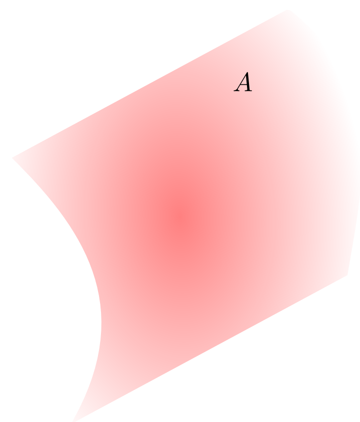

Figure 19: $T_X A + T_X B = T_X X$

Fundamental cycle

If M is an m - dimensional close, connected and orientable manifold, then $H_m(M, \mathbb{Z})$ and the orientation of M determined a cycle $[M] \in H_m(M, \mathbb{Z})$, called the fundamental cycle.

Example 6.1
If ω is an m - form then:

$$\int_M \omega = [\omega]([M]), \quad [\omega] \in H^m_\Omega(M), \; [M] \in H_m(M).$$

Example 6.2
Künneth ??????????????????????????

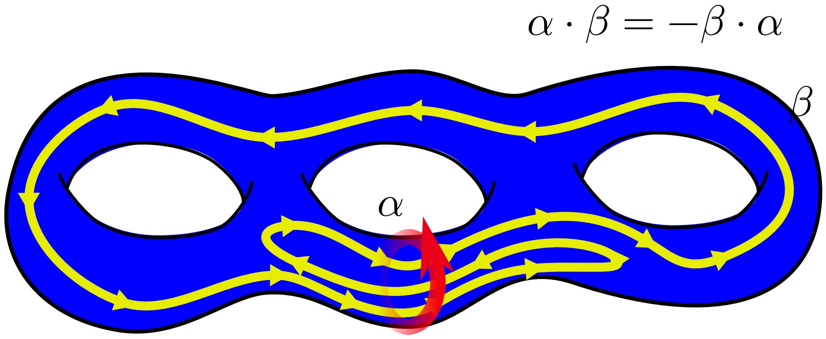

Figure 20: β cross 3 times the disk bounded by α. $T_X\alpha + T_X\beta = T_X\Sigma$

Let $X = S^2 \times S^2$. *We know that:*

$$H_2(S^2, \mathbb{Z}) = \mathbb{Z}$$
$$H_1(S^2, \mathbb{Z}) = 0$$
$$H_0(S^2, \mathbb{Z}) = \mathbb{Z}$$

We can construct a long exact sequence for a pair:

$$H_2(\partial X) \to H_2(X) \to H_2(X, \partial X) \to$$
$$\to H_1(\partial X) \to H_1(X) \to H_1(X, \partial X) \to$$

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Simple case $H_1(\partial X)$
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is torsion. $H_2(\partial X)$ *is torsion free (by universal coefficient theorem),*
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therefore it is 0.
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We know that $b_1(X) = b_2(X)$. *Therefore by Poincaré duality:*

$$b_1(X) = \dim_{\mathbb{Q}} H_1(X, \mathbb{Q}) \overset{\text{PD}}{=} \dim_{\mathbb{Q}} H^2(X, \mathbb{Q}) = \dim_{\mathbb{Q}} H_2(X, \mathbb{Q}) = b_2(X)$$

???????????????????????????????
$H_2(X, \mathbb{Z})$ *is torsion free and* $H_2(X_1, \mathbb{Q}) = 0$, *therefore* $H_2(X, \mathbb{Z}) = 0$. *The map* $H_2(X, \mathbb{Z}) \longrightarrow H_2(X, \partial X, \mathbb{Z})$ *is a monomorphism.*

??????????
(because it is an isomorphism after tensoring by $\mathbb{Q}$.
Suppose $\alpha_1, \dots, \alpha_n$ is a basis of $H_2(X, \mathbb{Z})$. Let A be the intersection matrix in this basis. Then:

1. *A has integer coefficients,*

2. *$\det A \neq 0$,*

3. *$|\det A| = |H_1(\partial X, \mathbb{Z})| = |\operatorname{coker} H_2(X) \longrightarrow H_2(X, \partial X)|$.*

???????????????????

If $CUC^T = W$, then for $\binom{a}{b} = C^{-1}\binom{1}{0}$ we have:

$$\begin{pmatrix} a \\ b \end{pmatrix} W \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} U \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \notin 2\mathbb{Z}.$$

Theorem 6.1 (Whitehead)
Any non-degenerate form

$$A : \mathbb{Z}^4 \times \mathbb{Z}^4 \longrightarrow \mathbb{Z}$$

can be realized as an intersection form of a simple connected 4-dimensional manifold.

??????????????????????????

Theorem 6.2 (Donaldson, 1982)
If A is an even definite intersection form of a smooth 4-manifold then it is diagonalizable over $\mathbb{Z}$.

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Definition 6.1
even define

Suppose X us 4 -manifold with a boundary such that $H_1(X) = 0$.

Proof. Obviously:

$$H_1(\partial X, \mathbb{Z}) = \operatorname{coker} H_2(X) \longrightarrow H_2(X, \partial X) = {H_2(X,\partial X)}\Big/{H_2(X)}.$$

Let A be an $n \times n$ matrix. A determines a
??????????????/

$$\begin{aligned} \mathbb{Z}^n &\longrightarrow \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}) \\ a &\mapsto (b \mapsto b^T A a) \\ |\operatorname{coker} A| &= |\det A| \end{aligned}$$

all homomorphisms $b = (b_1, \dots, b_n)$???????
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$\square$

Lecture 7 Linking form April 15, 2019

???????????????????

Theorem 7.1
Suppose that $K \subset S^3$ is a slice knot (i.e. K bound a disk in B^4). Then if F is a Seifert surface of K and V denotes the associated Seifet matrix, then there exists $P \in \mathrm{GL}_g(\mathbb{Z})$ such that:
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$$PVP^{-1} = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}, \quad A, C, C \in M_{g \times g}(\mathbb{Z}) \tag{1}$$

In other words you can find rank g direct summand $\mathcal{Z}$ of $H_1(F)$
????????????
such that for any $\alpha, \beta \in \mathcal{L}$ the linking number $\operatorname{lk}(\alpha, \beta^+) = 0$.

Definition 7.1
An abstract Seifert matrix (i. e.

Choose a basis $(b_1, ..., b_i)$
???
of $H_2(Y, \mathbb{Z}$, then $A = (b_i, b_y)$
??
is a matrix of intersection form:

$$\mathbb{Z}^n \big/ {A\mathbb{Z}^n} \cong H_1(Y, \mathbb{Z}).$$

In particular $|\det A| = \# H_1(Y, \mathbb{Z})$.
That means - what is happening on boundary is a measure of degeneracy.

$$\begin{array}{ccccc} H_1(Y, \mathbb{Z}) & \times & H_1(Y, \mathbb{Z}) & \longrightarrow & \mathbb{Q}\big/\mathbb{Z} \text{ - a linking form} \\ \wr\Vert & & \wr\Vert & & \\ \mathbb{Z}^n\big/A\mathbb{Z} & & \mathbb{Z}^n\big/A\mathbb{Z} & & \end{array}$$

$$(a, b) \mapsto aA^{-1}b^T$$

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The intersection form on a four-manifold determines the linking on the boundary.

Fact 7.1
Let $K \in S^1$ be a knot, $\Sigma(K)$ its double branched cover. If V is a Seifert matrix for K, then

$$H_1(\Sigma(K), \mathbb{Z}) \cong \mathbb{Z}^n \big/ A\mathbb{Z} \quad ,$$

where $A = V \times V^T$ and $n = \operatorname{rank} V$.

Let X be the four-manifold obtained via the double branched cover of B^4 branched along $\widetilde{\Sigma}$.

Fact 7.2

- *X is a smooth four-manifold,*

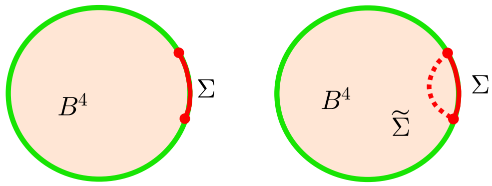


Figure 21: Pushing the Seifert surface in 4-ball.

- $H_1(X, \mathbb{Z}) = 0$,
- $H_2(X, \mathbb{Z}) \cong \mathbb{Z}^n$
- *The intersection form on X is $V + V^T$.*

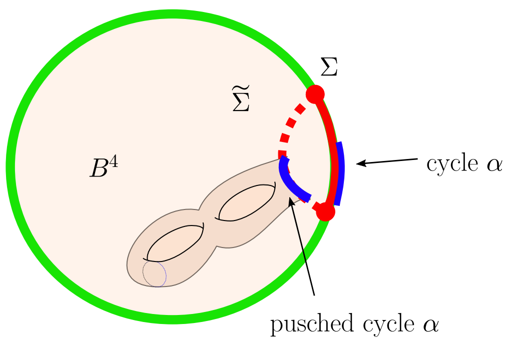


Figure 22: Cycle pushed in 4-ball.

Let $Y = \Sigma(K)$. Then:

$$H_1(Y, \mathbb{Z}) \times H_1(Y, \mathbb{Z}) \longrightarrow {}^{\mathbb{Q}}\!/_{\mathbb{Z}}$$
$$(a, b) \mapsto aA^{-1}b^T, \qquad A = V + V^T.$$

???????????????????????????

We have a primary decomposition of $H_1(Y, \mathbb{Z}) = U$ (as a group). For any $p \in \mathbb{P}$ we define U_p to be the subgroup of elements annihilated by the same power of p. We have $U = \bigoplus_p U_p$.

Example 7.1

$$\begin{aligned}
&\textit{If } U = \mathbb{Z}_3 \oplus \mathbb{Z}_{45} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{75} \textit{ then}\\
&U_3 = \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \textit{ and}\\
&U_5 = (e) \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{25}.
\end{aligned}$$

Lemma 7.1
Suppose $x \in U_{p_1}$, $y \in U_{p_2}$ and $p_1 \neq p_2$. Then $< x, y >= 0$.

Proof.

$$x \in U_{p_1}$$

□

$$H_1(Y, \mathbb{Z}) \cong {\mathbb{Z}^n}/{A\mathbb{Z}}$$
$$A \longrightarrow BAC^T \quad \text{Smith normal form}$$

??????????????????????
In general

Lecture 8 May 6, 2019

Definition 8.1
Let X be a knot complement. Then $H_1(X, \mathbb{Z}) \cong \mathbb{Z}$ and there exists an epimorphism $\pi_1(X) \overset{\phi}{\twoheadrightarrow} \mathbb{Z}$.
The infinite cyclic cover of a knot complement X is the cover associated with the epimorphism ϕ.

$$\widetilde{X} \longrightarrow\!\!\!\!\!\rightarrow X$$

Double branched cover.

Let $K \subset S^3$ be a knot and Σ its Seifert surface. Let us consider a knot complement $S^3 \setminus N(K)$. Formal sums $\sum \phi_i(t)a_i + \sum \phi_j(t)\alpha_j$
finitely generated as a $\mathbb{Z}[t, t^{-1}]$ module.
Let $v_{ij} = \mathrm{lk}(a_i, a_j^+)$. Then $V = \{v_ij\}_{i,j=1}^n$ is the Seifert matrix associated to the surface Σ and the basis $a_1, \dots, a_n$. Therefore $a_k^+ = \sum_j v_{jk}\alpha_j$.

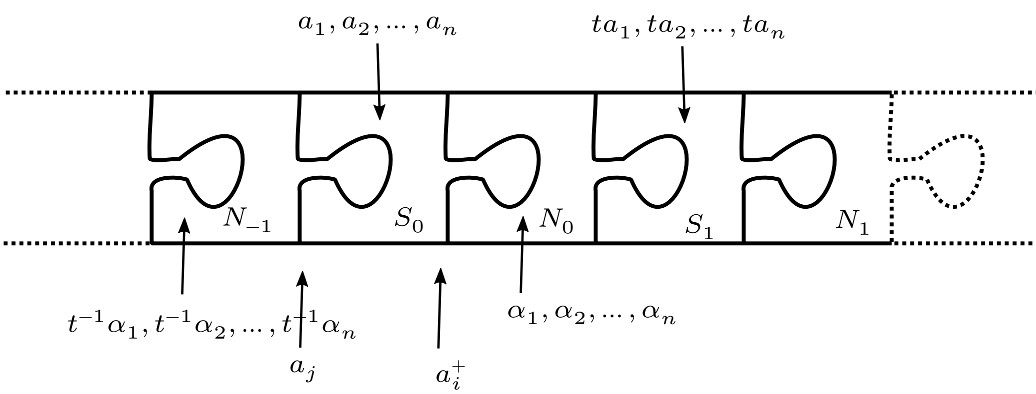


Figure 23: Infinite cyclic cover of a knot complement.

Then $\mathrm{lk}(a_i, a_k^+) = \mathrm{lk}(a_k^+, a_i) = \sum_j v_{jk} \,\mathrm{lk}(\alpha_j, a_i) = v_{ik}$. We also notice that $\mathrm{lk}(a_i, a_j^-) = \mathrm{lk}(a_i^+, a_j) = v_{ij}$ and $a_j^- = \sum_k v_{kj} t^{-1}\alpha_j$.
The homology of $\widetilde{X}$ is generated by $a_1, \dots, a_n$ and relations. Let now $H = H_1(\widetilde{X})$. Can we define a paring?
Let $c, d \in H(\widetilde{X})$ (see Figure 25), Δ an Alexander polynomial. We know that $\Delta c = 0 \in H_1(\widetilde{X})$ (Alexander polynomial annihilates all possible elements). Let consider a surface F such that $\partial F = c$. Now consider intersection points $F \cdot d$. This points can exist in any N_k or S_k.

$$\frac{1}{\Delta} \sum_{j \in \mathbb{Z} t^{-j}} (F \cdot t^j d) \in {}^{\mathbb{Q}[t, t^{-1}]} \big/ {}_{\mathbb{Z}[t, t^{-1}]}$$

?????????????
There is at least one paper where the structure of (Alexander module?) is calculated from a specific knot (?minimal number of generators?)
C. Kearton, S. M. J. Wilson

Fact 8.1
Let A be a matrix over principal ideal domain R. Than there exist matrices

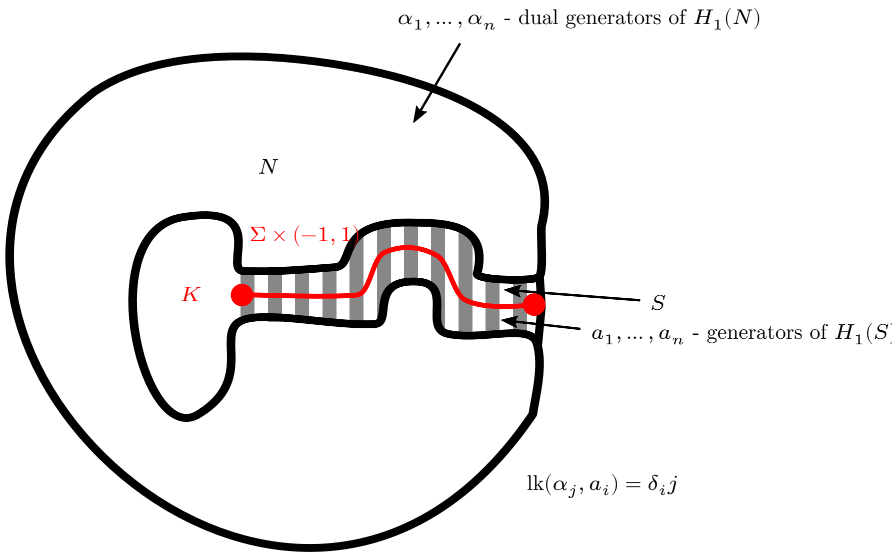


Figure 24: The double cover of the 3-sphere branched over a knot K.

C, D and E such that $A = CDE$,

$$D = \begin{bmatrix} d_1 & 0 & \cdots & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & d_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & d_n \end{bmatrix},$$

where $d_{i+1}|d_i$, and matrices C and E are invertible over R.
D is called a Smith normal form of the matrix A.

Definition 8.2
The $\mathbb{Z}[t, t^{-1}]$ module $H_1(\widetilde{X})$ is called the Alexander module of a knot K.

Let R be a PID, M a finitely generated R module. Let us consider

$$R^k \xrightarrow{A} R^n \twoheadrightarrow M,$$

where A is a $k \times n$ matrix, assume $k \geq n$. The order of M is the gcd of all determinants of the $n \times n$ minors of A. If $k = n$ then $\operatorname{ord} M = \det A$.

Theorem 8.1
Order of M doesn't depend on A.

For knots the order of the Alexander module is the Alexander polynomial.

Theorem 8.2

$$\forall x \in M : (\operatorname{ord} M)x = 0.$$

M is well defined up to a unit in R.
??????????????????
General picture : K, X knot complement...

$$\begin{aligned} H_1(X, \mathbb{Z}) &= \mathbb{Z} \\ H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) & \\ \pi_1(X) & \end{aligned}$$

Definition 8.3
The Nakanishi index of a knot is the minimal number of generators of $H_1(\widetilde{X})$.

Remark about notation: sometimes one writes $H_1(X; \mathbb{Z}[t, t^{-1}])$ (what is also notation for twisted homology) instead of $H_1(\widetilde{X})$.
?????????????????????
$\Sigma_?(K) \to S^3$?????
$H_1(\Sigma_?(K), \mathbb{Z}) = h$
$H \times H \longrightarrow {}^{\mathbb{Q}}\!/_{\mathbb{Z}}$
...

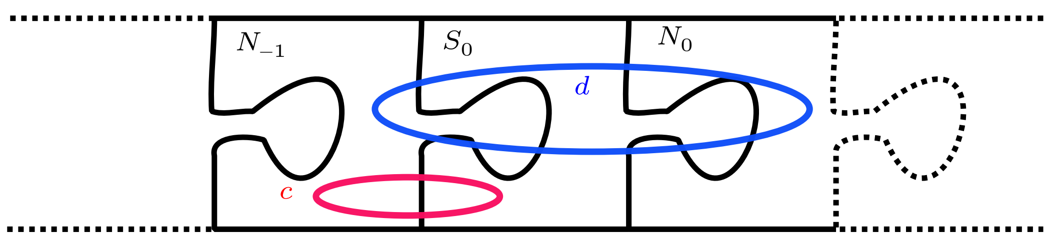


Figure 25: $c, d \in H_1(\widetilde{X})$.

Blanchfield pairing

Lecture 9 **May 20, 2019**

Let M be compact, oriented, connected four-dimensional manifold. If $H_1(M,\mathbb{Z}) = 0$ then there exists a bilinear form - the intersection form on M:

$$\begin{array}{ccccc} H_2(M,\mathbb{Z}) & \times & H_2(M,\mathbb{Z}) \longrightarrow & \mathbb{Z} \\ \wr\| & & & \\ \mathbb{Z}^n & & & \end{array}$$

Let us consider a specific case: M has a boundary $Y = \partial M$. Betti number $b_1(Y) = 0$, $H_1(Y,\mathbb{Z})$ is finite. Then the intersection form can be degenerated in the sense that:

$$\begin{array}{rr} H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \longrightarrow \mathbb{Z} & \qquad H_2(M,\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z}) \\ (a,b) \mapsto \mathbb{Z} & a \mapsto (a,\_) \in H_2(M,\mathbb{Z}) \end{array}$$

has coker precisely $H_1(Y,\mathbb{Z})$.
???????????????
Let $K \subset S^3$ be a knot, $X = S^3 \setminus K$ a knot complement and $\widetilde{X} \xrightarrow{\rho} X$ an infinite cyclic cover (universal abelian cover).
$C_*(\widetilde{X})$ has a structure of a $\mathbb{Z}[t,t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$ module.
Let $H_1(\widetilde{X},\mathbb{Z}[t,t^{-1}])$ be the Alexander module of the knot K with an intersection form:

$$H_1(\widetilde{X},\mathbb{Z}[t,t^{-1}]) \times H_1(\widetilde{X},\mathbb{Z}[t,t^{-1}]) \longrightarrow {}^{\mathbb{Q}}\!/_{\mathbb{Z}[t,t^{-1}]}$$

Fact 9.1

$$H_1(\widetilde{X},\mathbb{Z}[t,t^{-1}]) \cong {}^{\mathbb{Z}[t,t^{-1}]^n}\!/_{(tV - V^T)\mathbb{Z}[t,t^{-1}]^n},$$

where V is a Seifert matrix.

Fact 9.2

$$\begin{array}{rl} H_1(\widetilde{X},\mathbb{Z}[t,t^{-1}]) \times H_1(\widetilde{X},\mathbb{Z}[t,t^{-1}]) & \longrightarrow {}^{\mathbb{Q}}\!/_{\mathbb{Z}[t,t^{-1}]} \\ (\alpha,\beta) & \mapsto \alpha^{-1}(t-1)(tV-V^T)^{-1}\beta \end{array}$$

Note that $\mathbb{Z}[t,t^{-1}]$ is not PID. Therefore we don't have primary decomposition of this module. We can simplify this problem by replacing $\mathbb{Z}$ by $\mathbb{R}$. We lose some date by doing this transition, but we can

$$\begin{aligned} \xi \in S^1 \setminus \{\pm 1\} \quad & p_\xi = (t-\xi)(t-\xi^{-1})t^{-1} \\ \xi \in \mathbb{R} \setminus \{\pm 1\} \quad & q_\xi = (t-\xi)(t-\xi^{-1})t^{-1} \\ \xi \notin \mathbb{R} \cup S^1 \quad & q_\xi = (t-\xi)(t-\bar{\xi})(t-\xi^{-1})(t-\bar{\xi}^{-1})t^{-2} \end{aligned}$$

Let $\Lambda = \mathbb{R}[t,t^{-1}]$. Then:

$$H_1(\widetilde{X}, \Lambda) \cong \bigoplus_{\substack{\xi \in S^1 \setminus \{\pm 1\} \\ k \geq 0}} (\Lambda / p_\xi^k)^{n_k, \xi} \oplus \bigoplus_{\substack{\xi \notin S^1 \\ l \geq 0}} (\Lambda / q_\xi^l)^{n_l, \xi}$$

We can make this composition orthogonal with respect to the Blanchfield paring.

Historical remark:

- John Milnor, *On isometries of inner product spaces*, 1969,
- Walter Neumann, *Invariants of plane curve singularities* , 1983,
- András Némethi, *The real Seifert form and the spectral pairs of isolated hypersurfaceenumerate singularities*, 1995,
- Maciej Borodzik, Stefan Friedl *The unknotting number and classical invariants II*, 2014.

Let $p = p_\xi$, $k \geq 0$.

$$\begin{aligned} \Lambda / p^k\Lambda \times \Lambda / p^k\Lambda & \longrightarrow \mathbb{Q}(t) / \Lambda \\ (1,1) & \mapsto \kappa \\ \text{Now: } (p^k \cdot 1, 1) & \mapsto 0 \\ p^k \kappa = 0 & \in \mathbb{Q}(t) / \Lambda \\ \text{therfore } p^k \kappa & \in \Lambda \\ \text{we have } (1,1) & \mapsto \frac{h}{p^k} \end{aligned}$$

h is not uniquely defined: $h \to h + gp^k$ doesn't affect paring.
Let $h = p^k \kappa$.

Example 9.1

$$\phi_0((1,1)) = \frac{+1}{p}$$
$$\phi_1((1,1)) = \frac{-1}{p}$$

ϕ_0 and ϕ_1 are not isomorphic.

Proof. Let $\Phi : {}^{\Lambda}/_{p^k\Lambda} \longrightarrow {}^{\Lambda}/_{p^k\Lambda}$ be an isomorphism.
Let: $\Phi(1) = g \in \lambda$

$$\begin{array}{ccc} {}^{\Lambda}/_{p^k\Lambda} & \xrightarrow{\Phi} & {}^{\Lambda}/_{p^k\Lambda} \\ \phi_0((1,1)) = \dfrac{1}{p^k} & & \phi_1((g,g)) = \dfrac{1}{p^k} \quad (\Phi \text{ is an isometry}). \end{array}$$

Suppose for the paring $\phi_1((g,g)) = \frac{1}{p^k}$ we have $\phi_1((1,1)) = \frac{-1}{p^k}$. Then:

$$\frac{-g\bar{g}}{p^k} = \frac{1}{p^k} \in {}^{\mathbb{Q}(t)}/_{\Lambda}$$
$$\frac{-g\bar{g}}{p^k} - \frac{1}{p^k} \in \Lambda$$
$$-g\bar{g} \equiv 1 \pmod{p} \text{ in } \Lambda$$
$$-g\bar{g} - 1 = p^k\omega \text{ for some } \omega \in \Lambda$$
evalueting at ξ:
$$\overbrace{-g(\xi)g(\xi^{-1})}^{>0} - 1 = 0 \quad \Rightarrow\Leftarrow$$

$\square$

???????????????????

$$
\begin{aligned}
g &= \sum g_i t^i \\
\bar{g} &= \sum g_i t^{-i} \\
\bar{g}(\xi) &= \sum g_i \xi^i \quad \xi \in S^1 \\
\bar{g}(\xi) &= g(\bar{\xi})
\end{aligned}
$$

Suppose $g = (t-\xi)^{\alpha} g'$. Then $(t-\xi)^{k-\alpha}$ goes to 0 in ${}^{\Lambda}\!\big/_{p^k\Lambda}$.

Theorem 9.1

Every sesquilinear non-degenerate pairing

$$
{}^{\Lambda}\!\big/_{p^k} \times {}^{\Lambda}\!\big/_{p} \longrightarrow \frac{h}{p^k}
$$

is isomorphic either to the pairing wit $h = 1$ or to the paring with $h = -1$ depending on sign of $h(\xi)$ (which is a real number).

Proof. There are two steps of the proof:

1. Reduce to the case when h has a constant sign on S^1.

2. Prove in the case, when h has a constant sign on S^1.

Lemma 9.1

If P is a symmetric polynomial such that $P(\eta) \geq 0$ for all $\eta \in S^1$, then P can be written as a product $P = g\bar{g}$ for some polynomial g.

Sketch of proof. : Induction over $\deg P$.
Let $\zeta \notin S^1$ be a root of P, $P \in \mathbb{R}[t, t^{-1}]$. Assume $\zeta \notin \mathbb{R}$. We know that polynomial P is divisible by $(t-\zeta)$, $(t-\bar{\zeta})$, $(t^{-1}-\zeta)$ and $(t^{-1}-\bar{\zeta})$. Therefore:

$$
\begin{aligned}
P' &= \frac{P}{(t-\zeta)(t-\bar{\zeta})(t^{-1}-\zeta)(t^{-1}-\bar{\zeta})} \\
P' &= g'\bar{g}
\end{aligned}
$$

We set $g = g'(t-\zeta)(t-\bar{\zeta})$ and $P = g\bar{g}$. Suppose $\zeta \in S^1$. Then $(t-\zeta)^2|P$ (at least - otherwise it would change sign). Therefore:

$$
\begin{aligned}
P' &= \frac{P}{(t-\zeta)^2(t^{-1}-\zeta)^2} \\
g &= (t-\zeta)(t^{-1}-\zeta)g' \quad \text{etc.}
\end{aligned}
$$

The map $(1,1) \mapsto \frac{h}{p^k} = \frac{g\bar{g}h}{p^k}$ is isometric whenever g is coprime with P. $\square$

Lemma 9.2

Suppose A and B are two symmetric polynomials that are coprime and that $\forall z \in S^1$ either $A(z) > 0$ or $B(z) > 0$. Then there exist symmetric polynomials P, Q such that $P(z), Q(z) > 0$ for $z \in S^1$ and $PA + QB \equiv 1$.

Idea of proof. For any z find an interval (a_z, b_z) such that if $P(z) \in (a_z, b_z)$ and $P(z)A(z) + Q(z)B(z) = 1$, then $Q(z) > 0$, $x(z) = \frac{az+bz}{i}$ is a continues function on S^1 approximating z by a polynomial .
????????????????????????

$$(1,1) \mapsto \frac{h}{p^k} \mapsto \frac{g\bar{g}h}{p^k}$$
$$g\bar{g}h + p^k\omega = 1$$

Apply Lemma 9.2 for $A = h$, $B = p^{2k}$. Then, if the assumptions are satisfied,

$$Ph + Qp^{2k} = 1$$
$$p > 0 \Rightarrow p = g\bar{g}$$
$$p = (t-\xi)(t-\bar{\xi})t^{-1}$$
$$\text{so } p \geq 0 \text{ on } S^1$$
$$p(t) = 0 \Leftrightarrow t = \xi ort = \bar{\xi}$$
$$h(\xi) > 0$$
$$h(\bar{\xi}) > 0$$
$$g\bar{g}h + Qp^{2k} = 1$$
$$g\bar{g}h \equiv 1 \mod p^{2k}$$
$$g\bar{g} \equiv 1 \mod p^k$$

?????????????????????????????
If P has no roots on S^1 then $B(z) > 0$ for all z, so the assumptions of Lemma 9.2 are satisfied no matter what A is. □

????????????????

$$\Lambda\big/_{p_\xi^k} \times \Lambda\big/_{p_\xi^k} \longrightarrow \frac{\epsilon}{p_\xi^k}, \quad \xi \in S^1 \setminus \{\pm 1\}$$
$$\Lambda\big/_{q_\xi^k} \times \Lambda\big/_{q_\xi^k} \longrightarrow \frac{1}{q_\xi^k}, \quad \xi \notin S^1$$

?????????????????? 1 ?? epsilon?

Theorem 9.2 (Matumoto, Borodzik-Conway-Politarczyk)
Let K be a knot,

$$H_1(\widetilde{X}, \Lambda) \times H_1(\widetilde{X}, \Lambda) = \bigoplus_{\substack{k,\xi,\epsilon \\ \xi \in S^1}} (\Lambda\big/_{p_\xi^k}, \epsilon)^{n_k,\xi,\epsilon} \oplus \bigoplus_{k,\eta} (\Lambda\big/_{p_\xi^k})^{m_k} \text{ and}$$
$$\delta_\sigma(\xi) = \lim_{\varepsilon \to 0^+} \sigma(e^{2\pi i \varepsilon}\xi) - \sigma(e^{-2\pi i \varepsilon}\xi),$$
$$\text{then } \sigma_j(\xi) = \sigma(\xi) - \frac{1}{2} \lim_{\varepsilon \to 0} \sigma(e^{2\pi i \varepsilon}\xi) + \sigma(e^{-2\pi i \varepsilon}\xi)$$

The jump at ξ is equal to $2 \sum_{k_i \text{ odd}} \epsilon_i$.
The peak of the signature function is equal to $\sum_{k_i \text{ even}} \epsilon_i$.
?????????????????
$(\eta_{k,\xi_l^+} - \eta_{k,\xi_l^-}$

$\square$

Lecture 10 **May 27, 2019**

???????

Theorem 10.1
Such a pairing is isometric to a pairing:

$$[1] \times [1] \to \frac{\epsilon}{p_\xi^k},\ \epsilon \in \pm 1$$

?????????????

$$[1] = 1 \in {}^{\Lambda}\!/_{p_\xi^k \Lambda}$$

????????

Theorem 10.2
The jump of the signature function at ξ is equal to $2 \sum\limits_{k_i \text{ odd}} \epsilon_i$.
The peak of the signature function is equal to $\sum\limits_{k_i \text{ even}} \epsilon_i$.

$$({}^{\Lambda}\!/_{p^{k_1}\Lambda}, \epsilon_1) \oplus \cdots \oplus ({}^{\Lambda}\!/_{p^{k_n}\Lambda}, \epsilon_n)$$

Definition 10.1
A matrix A is called Hermitian if $\overline{A(t)} = A(t)^T$

Theorem 10.3 (Borodzik-Friedl 2015, Borodzik-Conway-Politarczyk 2018)
A square Hermitian matrix $A(t)$ of size n with coefficients in $\mathbb{Z}[t, t^{-1}]$ (or $\mathbb{R}[t, t^{-1}]$) represents the Blanchfield pairing if:

$$\begin{aligned} H_1(\bar{X}, \Lambda) &= {}^{\Lambda^n}\!/_{A\Lambda^n}, \\ (x, y) &\mapsto \overline{x}^T A^{-1} y \in {}^{\Omega}\!/_{\Lambda} \\ H_1(\widetilde{X}, \Lambda) \times H_1(\widetilde{X}, \Lambda) &\longrightarrow {}^{\Omega}\!/_{\Lambda}, \end{aligned}$$

where $\Lambda = \mathbb{Z}[t, t^{-1}]$ or $\mathbb{R}[t, t^{-1}]$, $\Omega = \mathbb{Q}(t)$ or $\mathbb{R}(t)$

????????
field of fractions ??????

$$\begin{aligned} H_1(\Sigma(K), \mathbb{Z}) &= {}^{\mathbb{Z}^n}\!/_{(V + V^T)\mathbb{Z}^n} \\ H_1(\Sigma(K), \mathbb{Z}) \times H_1(\Sigma(K), \mathbb{Z}) &\longrightarrow = {}^{\mathbb{Q}}\!/_{\mathbb{Z}} \\ (a, b) &\mapsto a(V + V^T)^{-1} b \end{aligned}$$

??????????????????
??????????????????

$$
\begin{aligned}
y &\mapsto y + Az \\
\overline{x^T} A^{-1}(y + Az) = \overline{x^T} A^{-1} y + \overline{x^T} \mathbb{1} z = \overline{x^T} A^{-1} y &\in {}^{\Omega}\big/_{\Lambda} \\
\overline{x^T} \mathbb{1} z &\in \Lambda \\
H_1(\widetilde{X}, \Lambda) &= {}^{\Lambda^n}\big/_{(Vt - V)\Lambda^n} \\
(a, b) &\mapsto \overline{a^T}(Vt - V^T)^{-1}(t-1)b
\end{aligned}
$$

(Blanchfield '59)

Theorem 10.4 (Kearton '75, Friedl, Powell '15)
There exits a matrix A representing the Blanchfield paring over $\mathbb{Z}[t, t^{-1}]$. The size of A is a size of Seifert form.

Remark:

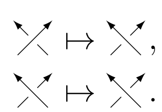
1. Over $\mathbb{R}$ we can take A to be diagonal.
2. The jump of signature function at ξ is equal to

$$\lim_{t \to 0^+} \operatorname{sign} A(e^{it}\xi) - \operatorname{sign} A(e^{-it}\xi).$$

3. The minimal size of a matrix A that presents a Blanchfield paring (over $\mathbb{Z}[t, t^{-1}]$) for a knot K is a knot invariant.

The unknotting number

Let K be a knot and D a knot diagram. A crossing change is a modification of a knot diagram by one of following changes



The unknotting number $u(K)$ is a number of crossing changes needed to turn a knot into an unknot, where the minimum is taken over all diagrams of a given knot.

Definition 10.2
A Gordian distance $G(K, K')$ between knots K and K' is the minimal number of crossing changes required to turn K into K'.

Problem 10.1
Prove that:

$$G(K, K'') \leq G(K, K') + G(K', K'').$$

Open problem:

$$u(K \# K') = u(K) + u(K').$$

Lemma 10.1 (Scharlemann '84)
Unknotting number one knots are prime.

Tools to bound unknotting number

Theorem 10.5
For any symmetric polynomial $\Delta \in \mathbb{Z}[t, t^{-1}]$ such that $\Delta(1) = 1$, there exists a knot K such that:

1. *K has unknotting number 1,*
2. *$\Delta_K = \Delta$.*

Let us consider a knot K and its Seifert surface Σ.
the Seifert form for K_-
the Seifert form for K_+
$S_- + S_+$ differs from by a term in the bottom right corner
Let A be a symmetric $n \times n$ matrix over $\mathbb{R}$. Let $A_1, \dots, A_n$ be minors of A.
Let $\epsilon_0 = 1$ If

Lecture 11 Surgery June 3, 2019

Theorem 11.1
Let K be a knot and $u(K)$ its unknotting number. Let g_4 be a minimal four

genus of a smooth surface S in B^4 such that $\partial S = K$. Then:

$$u(K) \geq g_4(K)$$

Proof. Recall that if $u(K) = u$ then K bounds a disk Δ with u ordinary double points.
???????????????

$$\begin{aligned} \chi(D^2) &= 1 \\ \chi(\Delta) &= 1 - u \\ \gamma = 0 &\in \pi_1(B^4 \setminus S) \end{aligned}$$

??????????????
Remove from Δ the two self intersecting disks and glue the Seifert surface for the Hopf link. The reality surface S has Euler characteristic $\chi(S) = 1 - 2u$. Therefore $g_4(S) = u$. □

Example 11.1
The knot 8_{20} is slice: $\sigma \equiv 0$ almost everywhere but $\sigma(e^{\frac{2\pi i}{6}}) = +1$.

Surgery

Recall that $H_1(S^1 \times S^1, \mathbb{Z}) = \mathbb{Z}^2$. As generators for H_1 we can set $\alpha = [S^1 \times \{\mathrm{pt}\}]$ and $\beta = [\{\mathrm{pt}\} \times S^1]$. Suppose $\phi : S^1 \times S^1 \longrightarrow S^1 \times S^1$ is a diffeomorphism. Consider an induced map on the homology group:

$$\begin{aligned} H_1(S^1 \times S^1, \mathbb{Z}) \ni \phi_*(\alpha) &= p\alpha + q\beta, \quad p, q \in \mathbb{Z}, \\ \phi_*(\beta) &= r\alpha + s\beta, \quad r, s \in \mathbb{Z}, \\ \phi_* &= \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \end{aligned}$$

As ϕ_* is diffeomorphis, it must be invertible over $\mathbb{Z}$. Then for a direction preserving diffeomorphism we have $\det \phi_* = 1$. *Therefore $\phi_* \in \mathrm{SL}(2, \mathbb{Z})$.*

Theorem 11.2
Every such a matrix can be realized as a torus.

Proof. (I) Geometric reason

$$\begin{aligned} \phi_t : S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ S^1 \times \{\mathrm{pt}\} &\longrightarrow \{\mathrm{pt}\} \times S^1 \\ \{\mathrm{pt}\} \times S^1 &\longrightarrow S^1 \times \{\mathrm{pt}\} \\ (x, y) &\mapsto (-y, x) \end{aligned}$$

(II)

□

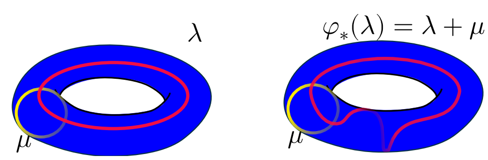


Figure 26: Dehn twist.

Sketch of proof. We will show that each diffeomorphism is isotopic to $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

$$\mathrm{Diff}_+(S^1 \times S^1) \big/ \mathrm{Iso}(S^1 \times S^1) = \mathrm{MCG}(S^1 \times S^1) = \mathrm{SL}(2, \mathbb{Z})$$

□

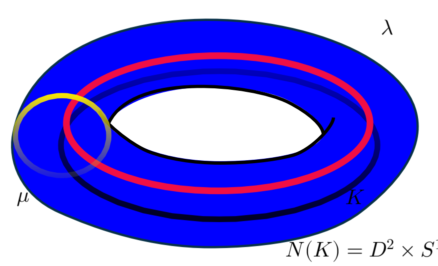


Figure 27: Choice of meridian and longitude.

Let $N = D^2 \times S$ be a tubular neighbourhood of a knot K. Consider its boundary $\partial N = S^1 \times S^1$. There exists a simple closed curve $\mu \subset \partial N$ (a meridian) that bounds a disk in N. We choose another simple closed curve λ (a longitude) so that $\mathrm{lk}(\lambda, K) = 0$.
????????
$\lambda\mu = 1$ intersection
$\pi_0(\mathrm{GL}(2, \mathbb{R})$
???????????
In other words a homotopy class: $[\lambda] = 0$ in $H_1(S^3 \setminus N, \mathbb{Z})$.

Lecture 12 Surgery — June 10, 2019

Consider a surgery

Lecture 13 Mess — June 17, 2019

Fact 13.1 (Milnor Singular Points of Complex Hypersurfaces)

An oriented knot is called negative amphichiral if the mirror image $m(K)$ of K is equivalent the reverse knot of K: K^r.

Problem 13.1
Prove that if K is negative amphichiral, then $K \# K = 0$ in $\mathcal{C}$.

Example 13.1
Figure 8 knot is negative amphichiral.

Theorem 13.1
Let H_p be a p - torsion part of H. There exists an orthogonal decomposition of H_p:

$$H_p = H_{p,1} \oplus \cdots \oplus H_{p,r_p}.$$

$H_{p,i}$ is a cyclic module:

$$H_{p,i} = {}^{\mathbb{Z}[t,t^{-1}]}\Big/_{p^{k_i}\mathbb{Z}[t,t^{-1}]}$$

The proof is the same as over $\mathbb{Z}$.