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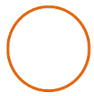
## Lecture 1 Basic definitions — February 25, 2019

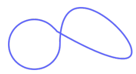
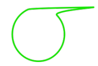
**Definition 1.1**
*A knot $K$ in $S^3$ is a smooth (PL - smooth) embedding of a circle $S^1$ in $S^3$:*

$$\varphi : S^1 \hookrightarrow S^3$$

Usually we think about a knot as an image of an embedding: $K = \varphi(S^1)$.

**Example 1.1**

- *Knots:*  *(unknot),*  *(trefoil).*

- *Not knots:*  *(it is not an injection),*  *(it is not smooth).*

**Definition 1.2**
*Two knots* $K_0 = \varphi_0(S^1)$*,* $K_1 = \varphi_1(S^1)$ *are equivalent if the embeddings* $\varphi_0$ *and* $\varphi_1$ *are isotopic, that is there exists a continues function*

$$\Phi : S^1 \times [0,1] \hookrightarrow S^3$$
$$\Phi(x,t) = \Phi_t(x)$$

*such that* $\Phi_t$ *is an embedding for any* $t \in [0,1]$*,* $\Phi_0 = \varphi_0$ *and* $\Phi_1 = \varphi_1$*.*

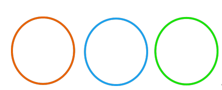
**Theorem 1.1**
*Two knots* $K_0$ *and* $K_1$ *are isotopic if and only if they are ambient isotopic, i.e. there exists a family of self-diffeomorphisms* $\Psi = \{\psi_t : t \in [0,1]\}$ *such that:*

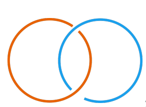
$$\psi(t) = \psi_t \text{ is continius on } t \in [0,1]$$
$$\psi_t : S^3 \hookrightarrow S^3,$$
$$\psi_0 = id,$$
$$\psi_1(K_0) = K_1.$$

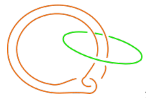
**Definition 1.3**
*A knot is trivial (unknot) if it is equivalent to an embedding* $\varphi(t) = (\cos t, \sin t, 0)$*, where* $t \in [0, 2\pi]$ *is a parametrisation of* $S^1$*.*

**Definition 1.4**
*A link with* $k$ *- components is a (smooth) embedding of* $\overbrace{S^1 \sqcup \ldots \sqcup S^1}^{k}$ *in* $S^3$

**Example 1.2**
*Links:*

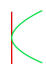
- *a trivial link with* 3 *components:* ,

- *a hopf link:* ,

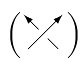
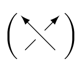
- *a Whitehead link:* ,
- *Borromean link:* ,

**Definition 1.5**
*A link diagram $D_\pi$ is a picture over projection $\pi$ of a link $L$ in $\mathbb{R}^3(S^3)$ to $\mathbb{R}^2$ $(S^2)$ such that:*

*(1) $D_{\pi|_L}$ is non degenerate:* ,

*(2) the double points are not degenerate:* ,

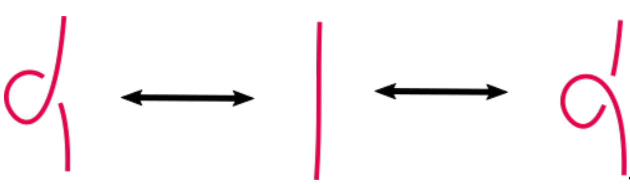
*(3) there are no triple point:* .

There are under- and overcrossings (tunnels and bridges) on a link diagrams with an obvious meaning.
Every link admits a link diagram.
Let $D$ be a diagram of an oriented link (to each component of a link we add an arrow in the diagram).
We can distinguish two types of crossings: right-handed , called a positive crossing, and left-handed , called a negative crossing.

## 1.1 Reidemeister moves

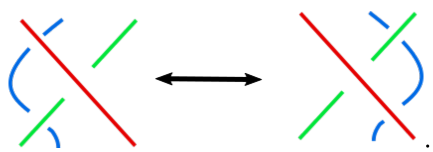
A Reidemeister move is one of the three types of operation on a link diagram as shown below:

I

,

II

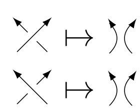
III



**Theorem 1.2** (Reidemeister, 1927 )
*Two diagrams of the same link can be deformed into each other by a finite sequence of Reidemeister moves (and isotopy of the plane).*

## 1.2 Seifert surface

Let $D$ be an oriented diagram of a link $L$. We change the diagram by smoothing each crossing:



We smooth all the crossings, so we get a disjoint union of circles on the plane. Each circle bounds a disks in $\mathbb{R}^3$ (we choose disks that don't intersect). For each smoothed crossing we add a twisted band: right-handed for a positive and left-handed for a negative one. We get an orientable surface $\Sigma$ such that $\partial\Sigma = L$.

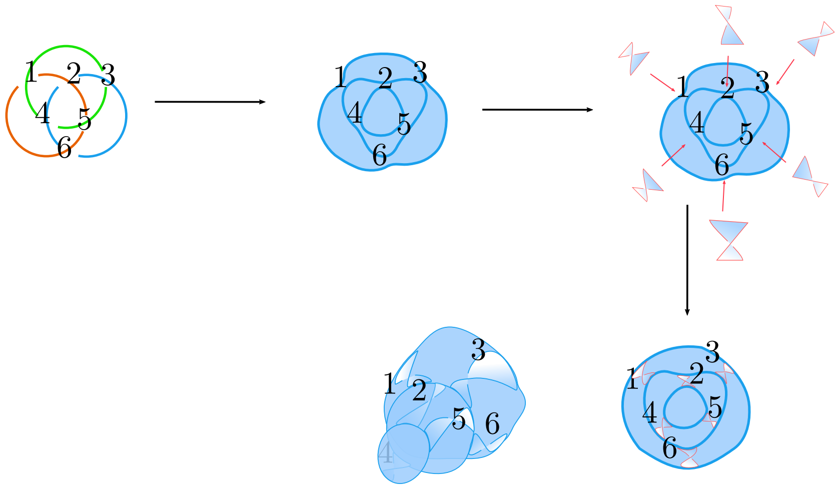



Figure 1: Constructing a Seifert surface.

Note: in general the obtained surface doesn't need to be connected, but by taking connected sum of all components we can easily get a connected surface (i.e. we take two disconnected components and cut a disk in each of them: $D_1$ and $D_2$; now we glue both components on the boundaries: $\partial D_1$ and $\partial D_2$.

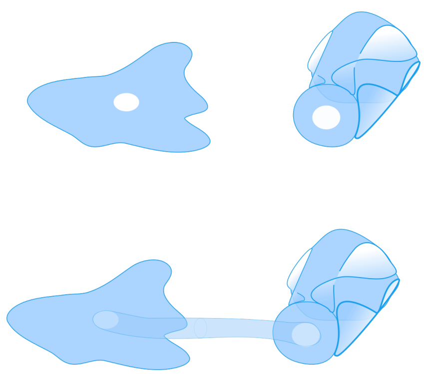


Figure 2: Connecting two surfaces.

**Theorem 1.3** (Seifert)
*Every link in $S^3$ bounds a surface $\Sigma$ that is compact, connected and orientable. Such a surface is called a Seifert surface.*

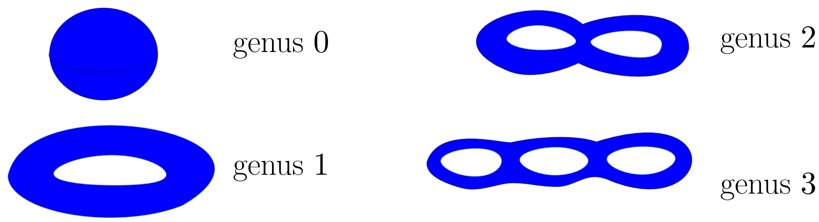



Figure 3: Genus of an orientable surface.

**Definition 1.6**
*The three genus $g_3(K)$ $(g(K))$ of a knot $K$ is the minimal genus of a Seifert surface $\Sigma$ for $K$.*

**Corollary 1.1**
*A knot $K$ is trivial if and only $g_3(K) = 0$.*

Remark: there are knots that admit non isotopic Seifert surfaces of minimal genus (András Juhász, 2008).

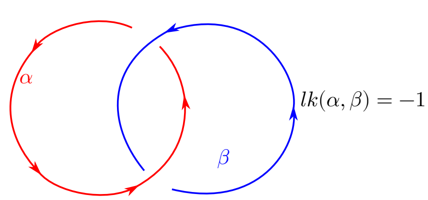
**Definition 1.7**
*Suppose $\alpha$ and $\beta$ are two simple closed curves in $\mathbb{R}^3$. On a diagram $L$ consider all crossings between $\alpha$ and $\beta$. Let $N_+$ be the number of positive crossings, $N_-$ - negative. Then the linking number: $lk(\alpha, \beta) = \frac{1}{2}(N_+ - N_-)$.*

Let $\alpha$ and $\beta$ be two disjoint simple cross curves in $S^3$. Let $\nu(\beta)$ be a tubular neighbourhood of $\beta$. The linking number can be interpreted via first homology group, where $lk(\alpha, \beta)$ is equal to evaluation of $\alpha$ as element of first homology group of the complement of $\beta$:

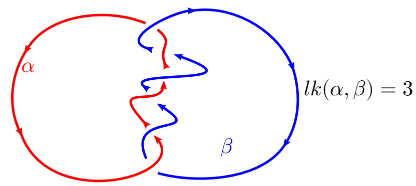
$$\alpha \in H_1(S^3 \setminus \nu(\beta), \mathbb{Z}) \cong \mathbb{Z}.$$

**Example 1.3**

- *Hopf link*

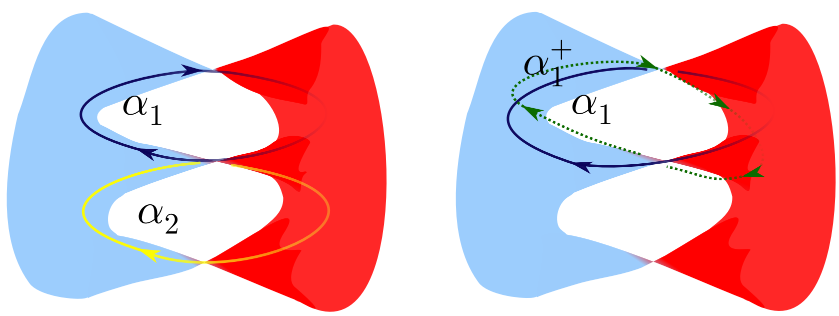


- $T(6,2)$ *link*




## 1.3 Seifert matrix

Let $L$ be a link and $\Sigma$ be an oriented Seifert surface for $L$. Choose a basis for $H_1(\Sigma, \mathbb{Z})$ consisting of simple closed $\alpha_1, \ldots, \alpha_n$. Let $\alpha_1^+, \ldots \alpha_n^+$ be copies of $\alpha_i$ lifted up off the surface (push up along a vector field normal to $\Sigma$). Note that elements $\alpha_i$ are contained in the Seifert surface while all $\alpha_i^+$ are don't intersect the surface. Let $lk(\alpha_i, \alpha_j^+) = \{a_{ij}\}$. Then the matrix $S = \{a_{ij}\}_{i,j=1}^n$ is called a Seifert matrix for $L$. Note that by choosing a different basis we get a different matrix.




**Theorem 1.4**
*The Seifert matrices $S_1$ and $S_2$ for the same link $L$ are S-equivalent, that is, $S_2$ can be obtained from $S_1$ by a sequence of following moves:*

*(1)* $V \to AVA^T$, *where $A$ is a matrix with integer coefficients,*

*(2)* $V \to \left(\begin{array}{ccc|cc} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \ldots & * & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{array}\right)$ *or* $V \to \left(\begin{array}{ccc|cc} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \ldots & * & 0 & 1 \\ 0 & \ldots & 0 & 0 & 0 \end{array}\right)$

*(3) inverse of (2)*

**Theorem 2.1**
*For any knot $K \subset S^3$ there exists a connected, compact and orientable surface $\Sigma(K)$ such that $\partial\Sigma(K) = K$*

*Proof.* ("joke")
Let $K \in S^3$ be a knot and $N = \nu(K)$ be its tubular neighbourhood. Because $K$ and $N$ are homotopy equivalent, we get:

$$H^1(S^3 \setminus N) \cong H^1(S^3 \setminus K).$$

Let us consider a long exact sequence of cohomology of a pair $(S^3, S^3 \setminus N)$ with integer coefficients:

$$\begin{array}{ccc}
 & \mathbb{Z} & \\
 & \wr\| & \\
 & H^0(S^3) \to & H^0(S^3 \setminus N) \to \\
 & & \\
\to H^1(S^3, S^3 \setminus N) \to & H^1(S^3) \to & H^1(S^3 \setminus N) \to \\
 & \wr\| & \\
 & 0 & \\
 & \wr\| & \\
\to H^2(S^3, S^3 \setminus N) \to & H^2(S^3) \to & H^2(S^3 \setminus N) \to \\
 & & \\
\to H^3(S^3, S^3 \setminus N) \to & H^3(S) \to & 0 \\
 & \wr\| & \\
 & \mathbb{Z} &
\end{array}$$

$$H^*(S^3, S^3 \setminus N) \cong H^*(N, \partial N)$$

?????????????

$\square$

**Definition 2.1**
*Let $S$ be a Seifert matrix for a knot $K$. The Alexander polynomial $\Delta_K(t)$ is a Laurent polynomial:*

$$\Delta_K(t) := \det(tS - S^T) \in \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$$

**Theorem 2.2**
*$\Delta_K(t)$ is well defined up to multiplication by $\pm t^k$, for $k \in \mathbb{Z}$.*

*Proof.* We need to show that $\Delta_K(t)$ doesn't depend on $S$-equivalence relation.

(1) Suppose $S' = CSC^T$, $C \in \mathrm{Gl}(n, \mathbb{Z})$ (matrices invertible over $\mathbb{Z}$). Then $\det C = 1$ and:

$$\det(tS' - S'^T) = \det(tCSC^T - (CSC^T)^T) =$$
$$\det(tCSC^T - CS^TC^T) = \det C(tS - S^T)C^T = \det(tS - S^T)$$

(2) Let

$$A := t\left(\begin{array}{ccc|cc} & S & & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right) - \left(\begin{array}{ccc|cc} & S^T & & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} & tS - S^T & & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & -1 \\ 0 & \dots & 0 & t & 0 \end{array}\right)$$

Using the Laplace expansion we get $\det A = \pm t \det(tS - S^T)$.

□

**Example 2.1**
*If $K$ is a trefoil then we can take $S = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$.*

$\Delta_K(t) = \det \begin{pmatrix} -t+1 & -t \\ 1 & -t+1 \end{pmatrix} = (t-1)^2 + t = t^2 - t + 1 \neq 1 \Rightarrow$ *trefoil is not trivial*

**Fact 2.1**
*$\Delta_K(t)$ is symmetric.*

*Proof.* Let $S$ be an $n \times n$ matrix.

$$\Delta_K(t^{-1}) = \det(t^{-1}S - S^T) = (-t)^{-n} \det(tS^T - S) =$$
$$(-t)^{-n} \det(tS - S^T) = (-t)^{-n} \Delta_K(t)$$

If $K$ is a knot, then $n$ is necessarily even, and so $\Delta_K(t^{-1}) = t^{-n} \Delta_K(t)$. □

**Lemma 2.1**

$$\frac{1}{2}\deg \Delta_K(t) \leq g_3(K),\ \textit{where } deg(a_n t^n + \cdots + a_1 t^l) = k - l.$$

*Proof.* If $\Sigma$ is a genus $g$ - Seifert surface for $K$ then $H_1(\Sigma) = \mathbb{Z}^{2g}$, so $S$ is an $2g \times 2g$ matrix. Therefore $\det(tS - S^T)$ is a polynomial of degree at most $2g$. □

**Example 2.2**
*There are not trivial knots with Alexander polynomial equal* 1*, for example:* $\Delta_{11n34} \equiv 1$.

**Lecture 3**

**Example 3.1**

$$F : \mathbb{C}^2 \to \mathbb{C} \ \textit{a polynomial}$$
$$F(0) = 0$$

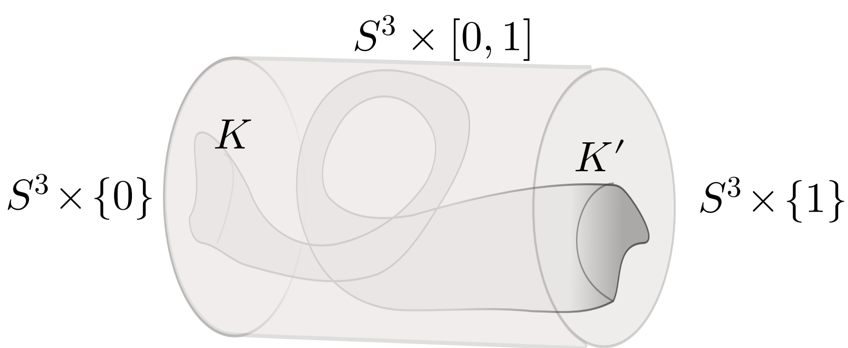
*Fact (Milnor Singular Points of Complex Hypersurfaces):*

An oriented knot is called negative amphichiral if the mirror image $m(K)$ if $K$ is equivalent the reverse knot of $K$.

**Example 3.2** (Problem)
*Prove that if* $K$ *is negative amphichiral, then* $K\#K$ *in* **C**

**Definition 4.1**
*A knot $K$ is called (smoothly) slice if $K$ is smoothly concordant to an unknot. A knot $K$ is smoothly slice if and only if $K$ bounds a smoothly embedded disk in $B^4$.*

**Definition 4.2**
*Two knots $K$ and $K'$ are called (smoothly) concordant if there exists an annulus $A$ that is smoothly embedded in $S^3 \times [0,1]$ such that $\partial A = K' \times \{1\} \sqcup K \times \{0\}$.*




Let $m(K)$ denote a mirror image of a knot $K$.

**Fact 4.1**
*For any $K$, $K\#m(K)$ is slice.*

**Fact 4.2**
*Concordance is an equivalence relation.*

**Fact 4.3**
*If $K_1 \sim {K_1}'$ and $K_2 \sim {K_2}'$, then $K_1\#K_2 \sim {K_1}'\#{K_2}'$.*

**Fact 4.4**
*$K\#m(K) \sim$ the unknot.*

Let $\mathcal{C}$ denote all equivalent classes for knots. $\mathcal{C}$ is a group under taking connected sums, with neutral element (the class defined by) an unknot and inverse element (a class defined by) a mirror image.
The figure eight knot is a torsion element in $\mathcal{C}$ ($2K \sim$ the unknot).

**Example 4.1** (Problem)
*Are there in concordance group torsion elements that are not 2 torsion elements? (open)*

Remark: $K \sim K' \Leftrightarrow K\# - K'$ is slice.

**Lecture 5** **April 8, 2019**

$X$ is a closed orientable four-manifold. Assume $\pi_1(X) = 0$ (it is not needed to define the intersection form). In particular $H_1(X) = 0$. $H_2$ is free (exercise).

$$H_2(X, \mathbb{Z}) \xrightarrow{\text{Poincaré duality}} H^2(X, \mathbb{Z}) \xrightarrow{\text{evaluation}} \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$$

Intersection form: $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ - symmetric, non singular.
Let $A$ and $B$ be closed, oriented surfaces in $X$.

**Proposition 5.1**
*$A \cdot B$ doesn't depend of choice of $A$ and $B$ in their homology classes.*

**Lecture 6** **April 15, 2019**

In other words:
Choose a basis $(b_1, ..., b_i)$
???
of $H_2(Y, \mathbb{Z}$, then $A = (b_i, b_y)$
??
is a matrix of intersection form:

$$\mathbb{Z}^n \big/ {A\mathbb{Z}^n} \cong H_1(Y, \mathbb{Z}).$$

In particular $|\det A| = \# H_1(Y, \mathbb{Z}$.
That means - what is happening on boundary is a measure of degeneracy.

$$\begin{array}{ccccl} H_1(Y,\mathbb{Z}) & \times & H_1(Y,\mathbb{Z}) & \longrightarrow & {}^{\mathbb{Q}}\!/_{\mathbb{Z}} \text{ - a linking form} \\ \wr\Vert & & \wr\Vert & & \\ {}^{\mathbb{Z}^n}\!/_{A\mathbb{Z}} & & {}^{\mathbb{Z}^n}\!/_{A\mathbb{Z}} & & \end{array}$$

$$(a, b) \mapsto aA^{-1}b^T$$

The intersection form on a four-manifold determines the linking on the boundary.

Let $K \in S^1$ be a knot, $\Sigma(K)$ its double branched cover. If $V$ is a Seifert matrix for $K$, then $H_1(\Sigma(K), \mathbb{Z}) \cong {}^{\mathbb{Z}^n}\!/_{A\mathbb{Z}}$ where $A = V \times V^T$, where $n = \operatorname{rank} V$.

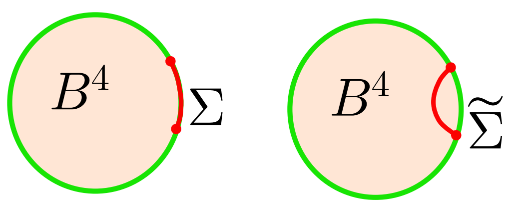



Figure 4: Pushing the Seifert surface in 4-ball.

Let $X$ be the four-manifold obtained via the double branched cover of $B^4$ branched along $\widetilde{\Sigma}$.

**Fact 6.1**

- *$X$ is a smooth four-manifold,*
- *$H_1(X, \mathbb{Z}) = 0$,*
- *$H_2(X, \mathbb{Z}) \cong \mathbb{Z}^n$*
- *The intersection form on $X$ is $V + V^T$.*

Let $Y = \Sigma(K)$. Then:

$$H_1(Y,\mathbb{Z}) \times H_1(Y,\mathbb{Z}) \longrightarrow \mathbb{Q}/_{\mathbb{Z}}$$
$$(a,b) \mapsto aA^{-1}b^T, \qquad A = V + V^T$$
$$H_1(Y,\mathbb{Z}) \cong \mathbb{Z}^n/_{A\mathbb{Z}}$$
$$A \longrightarrow BAC^T \quad \text{Smith normal form}$$

??????????????????????
In general

**Lecture 7** **May 20, 2019**

Let $M$ be compact, oriented, connected four-dimensional manifold. If $H_1(M,\mathbb{Z}) = 0$ then there exists a bilinear form - the intersection form on $M$:

$$\begin{matrix} H_2(M,\mathbb{Z}) & \times & H_2(M,\mathbb{Z}) \longrightarrow & \mathbb{Z} \\ \wr\| & & & \\ \mathbb{Z}^n & & & \end{matrix}$$

Let us consider a specific case: $M$ has a boundary $Y = \partial M$.
Betti number $b_1(Y) = 0$, $H_1(Y,\mathbb{Z})$ is finite.
Then the intersection form can be degenerate in the sense that

$$H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \longrightarrow \mathbb{Z} \qquad H_2(M,\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z})$$
$$(a,b) \mapsto \mathbb{Z} \qquad a \mapsto (a,\_)H_2(M,\mathbb{Z})$$

has coker precisely $H_1(Y,\mathbb{Z})$.
???????????????
Let $K \subset S^3$ be a knot,

$X = S^3 \setminus K$ - a knot complement,
$\widetilde{X} \xrightarrow{\rho} X$ - an infinite cyclic cover (universal abelian cover).

$$\pi_1(X) \longrightarrow {}^{\pi_1(X)}\!\big/_{[\pi_1(X), \pi_1(X)]} = H_1(X, \mathbb{Z}) \cong \mathbb{Z}$$

$C_*(\widetilde{X})$ has a structure of a $\mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$ module.
$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}])$ - Alexander module,

$$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \times H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \longrightarrow {}^{\mathbb{Q}}\!\big/_{\mathbb{Z}[t, t^{-1}]}$$

**Fact 7.1**

$$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \cong {}^{\mathbb{Z}[t, t^{-1}]^n}\!\big/_{(tV - V^T)\mathbb{Z}[t, t^{-1}]^n},$$

*where $V$ is a Seifert matrix.*

**Fact 7.2**

$$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \times H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \longrightarrow {}^{\mathbb{Q}[t]}\!\big/_{\mathbb{Z}[t, t^{-1}]}$$
$$(\alpha, \beta) \mapsto \alpha^{-1}(t-1)(tV - V^T)^{-1}\beta$$

Note that $\mathbb{Z}$ is not PID. Therefore we don't have primer decomposition of this module. We can simplify this problem by replacing $\mathbb{Z}$ by $\mathbb{R}$. We lose some date by doing this transition.

$\xi \in S^1 \setminus \{\pm 1\} \quad p_\xi = (t - \xi)(t - \xi^{-1})t^{-1}$
$\xi \in \mathbb{R} \setminus \{\pm 1\} \quad q_\xi = (t - \xi)(t - \xi^{-1})t^{-1}$
$\xi \notin \mathbb{R} \cup S^1 \quad q_\xi = (t - \xi)(t - \bar{\xi})(t - \xi^{-1})(1 - \bar{\xi}^{-1})t^{-2}$
$\Lambda = \mathbb{R}[t, t^{-1}]$
Then: $H_1(\widetilde{X}, \Lambda) \cong \bigoplus_{\substack{\xi \in S^1 \setminus \{\pm 1\} \\ k \geq 0}} ({}^{\Lambda}\!/_{p_\xi^k})^{n_{k,\xi}} \oplus \bigoplus_{\substack{\xi \notin S^1 \\ l \geq 0}} ({}^{\Lambda}\!/_{q_\xi^l})^{n_{l,\xi}}$

We can make this composition orthogonal with respect to the Blanchfield paring.

Historical remark:

- John Milnor, *On isometries of inner product spaces*, 1969,
- Walter Neumann, *Invariants of plane curve singularities* , 1983,
- András Némethi, *The real Seifert form and the spectral pairs of isolated hypersurfaceenumerate singularities*, 1995,
- Maciej Borodzik, Stefan Friedl *The unknotting number and classical invariants II*, 2014.

Let $p = p_\xi$, $k \geq 0$.

$$\Lambda/_{p^k\Lambda} \times \Lambda/_{p^k\Lambda} \longrightarrow \mathbb{Q}(t)/_{\Lambda}$$
$$(1,1) \mapsto \kappa$$
$$\text{Now: } (p^k \cdot 1, 1) \mapsto 0$$
$$p^k\kappa = 0 \in \mathbb{Q}(t)/_{\Lambda}$$
$$\text{therfore } p^k\kappa \in \Lambda$$
$$\text{we have } (1,1) \mapsto \frac{h}{p^k}$$

$h$ is not uniquely defined: $h \to h + gp^k$ doesn't affect paring.
Let $h = p^k\kappa$.

**Example 7.1**

$$\phi_0((1,1)) = \frac{+1}{p}$$
$$\phi_1((1,1)) = \frac{-1}{p}$$

*$\phi_0$ and $\phi_1$ are not isomorphic.*

*Proof.* Let $\Phi : \Lambda/_{p^k\Lambda} \longrightarrow \Lambda/_{p^k\Lambda}$ be an isomorphism.
Let: $\Phi(1) = g \in \lambda$

$$\Lambda/_{p^k\Lambda} \xrightarrow{\Phi} \Lambda/_{p^k\Lambda}$$
$$\phi_0((1,1)) = \frac{1}{p^k} \qquad \phi_1((g,g)) = \frac{1}{p^k} \quad (\Phi \text{ is an isometry}).$$

Suppose for the paring $\phi_1((g,g)) = \frac{1}{p^k}$ we have $\phi_1((1,1)) = \frac{-1}{p^k}$. Then:

$$\begin{aligned} \frac{-g\bar{g}}{p^k} &= \frac{1}{p^k} \in {}^{\mathbb{Q}(t)}\big/_{\Lambda} \\ \frac{-g\bar{g}}{p^k} - \frac{1}{p^k} &\in \Lambda \\ -g\bar{g} &\equiv 1 \pmod{p} \text{ in } \Lambda \\ -g\bar{g} - 1 &= p^k\omega \text{ for some } \omega \in \Lambda \\ \text{evalueting at } \xi: \\ \overbrace{-g(\xi)g(\xi^{-1})}^{>0} -1 = 0 \quad &\Rightarrow\Leftarrow \end{aligned}$$

□

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$$\begin{aligned} g &= \sum g_i t^i \\ \bar{g} &= \sum g_i t^{-i} \\ \bar{g}(\xi) &= \sum g_i \xi^i \quad \xi \in S^1 \\ \bar{g}(\xi) &= g(\bar{\xi}) \end{aligned}$$

Suppose $g = (t-\xi)^\alpha g'$. Then $(t-\xi)^{k-\alpha}$ goes to 0 in ${}^{\Lambda}\big/_{p^k\Lambda}$.

**Theorem 7.1**
*Every sesquilinear non-degenerate pairing*

$${}^{\Lambda}\big/_{p^k} \times {}^{\Lambda}\big/_{p} \longleftrightarrow \frac{h}{p^k}$$

*is isomorphic either to the pairing wit $h = 1$ or to the paring with $h = -1$ depending on sign of $h(\xi)$ (which is a real number).*

*Proof.* There are two steps of the proof:

1. Reduce to the case when $h$ has a constant sign on $S^1$.

2. Prove in the case, when $h$ has a constant sign on $S^1$.

**Lemma 7.1**
*If $p$ is a symmetric polynomial such that$p(\eta) \geq 0$ for all $\eta \in S^1$, then $p$ can be written as a product $p = g\bar{g}$ for some polynomial $g$.*

*Sketch of proof.* Induction over $\deg p$.
Let $\zeta \notin S^1$ be a root of $p$, $p \in \mathbb{R}[t, t^{-1}]$. Assume $\zeta \notin \mathbb{R}$. We know that

$$\begin{aligned}(t-\zeta) &\mid p,\\ (t-\bar{\zeta}) &\mid p,\\ (t^{-1}-\zeta) &\mid p,\\ (t^{-1}-\bar{\zeta}) &\mid p,\end{aligned}$$

therefore:

$$\begin{aligned}p' = \frac{p}{(t-\zeta)(t-\bar{\zeta})(t^{-1}-\zeta)(t^{-1}-\bar{\zeta})}\\ p' = g'\bar{g}\\ \text{we set } g = g'(t-\zeta)(t-\bar{\zeta}\\ p = g\bar{g}\end{aligned}$$

Suppose $\zeta \in S^1$. Then $(t-\zeta)^2 \mid p$ (at least - otherwise it would change sign).

$$\begin{aligned}p' = \frac{p}{(t-\zeta)^2(t^{-1}-\zeta)^2}\\ g = (t-\zeta)(t^{-1}-\zeta)g' \quad \text{etc.}\end{aligned}$$

$(1,1) \mapsto \dfrac{h}{p^k} = \dfrac{g\bar{g}h}{p^k}$ isometry whenever $g$ is coprime with $p$.

□

**Lemma 7.2**
*Suppose $A$ and $B$ are two symmetric polynomials that are coprime and that $\forall z \in S^1$ either $A(z) > 0$ or $B(z) > 0$. Then there exist symmetric polynomials $P$, $Q$ such that $P(z), Q(z) > 0$ for $z \in S^1$ and $PA + QB \equiv 1$.*

*Idea of proof.* For any $z$ find an interval $(a_z, b_z)$ such that if $P(z) \in (a_z, b_z)$ and $P(z)A(z) + Q(z)B(z) = 1$, then $Q(z) > 0$, $x(z) = \frac{az+bz}{i}$ is a continues function on $S^1$ approximating $z$ by a polynomial .

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$$(1,1) \mapsto \frac{h}{p^k} \mapsto \frac{g\bar{g}h}{p^k}$$
$$g\bar{g}h + p^k\omega = 1$$

Apply Lemma 7.2 for $A = h$, $B = p^{2k}$. Then, if the assumptions are satisfied,

$$Ph + Qp^{2k} = 1$$
$$p > 0 \Rightarrow p = g\bar{g}$$
$$p = (t-\xi)(t-\bar{\xi})t^{-1}$$
$$\text{so } p \geq 0 \text{ on } S^1$$
$$p(t) = 0 \Leftrightarrow t = \xi ort = \bar{\xi}$$
$$h(\xi) > 0$$
$$h(\bar{\xi}) > 0$$
$$g\bar{g}h + Qp^{2k} = 1$$
$$g\bar{g}h \equiv 1 \mod p^{2k}$$
$$g\bar{g} \equiv 1 \mod p^k$$

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If $P$ has no roots on $S^1$ then $B(z) > 0$ for all $z$, so the assumptions of Lemma 7.2 are satisfied no matter what $A$ is. □

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$$({}^{\Lambda}\!/_{p_\xi^k} \times {}^{\Lambda}\!/_{p_\xi^k}) \longrightarrow \frac{\epsilon}{p_\xi^k}, \quad \xi \in S^1 \setminus \{\pm 1\}$$
$$({}^{\Lambda}\!/_{q_\xi^k} \times {}^{\Lambda}\!/_{q_\xi^k}) \longrightarrow \frac{1}{q_\xi^k}, \quad \xi \notin S^1$$

??????????????????? 1 ?? epsilon?

**Theorem 7.2**

*(Matumoto, Conway-Borodzik-Politarczyk) Let $K$ be a knot,*

$$H_1(\widetilde{X}, \Lambda) \times H_1(\widetilde{X}, \Lambda) = \bigoplus_{\substack{k,\xi,\epsilon \\ \xi in S^1}} ({}^{\Lambda}/_{p_\xi^k}, \epsilon)^{n_k,\xi,\epsilon} \oplus \bigoplus_{k,\eta} ({}^{\Lambda}/_{p_\xi^k})^{m_k}$$

$$\text{Let } \delta_\sigma(\xi) = \lim_{\varepsilon \to 0^+} \sigma(e^{2\pi i \varepsilon}\xi) - \sigma(e^{-2\pi i \varepsilon}\xi),$$
$$\text{then } \sigma_j(\xi) = \sigma(\xi) - \frac{1}{2} \lim_{\varepsilon \to 0} \sigma(e^{2\pi i \varepsilon}\xi) + \sigma(e^{-2\pi i \varepsilon}\xi)$$

*The jump at $\xi$ is equal to $2 \sum_{k_i \ odd} \epsilon_i$. The peak of the signature function is equal to $\sum_{k_i even} \epsilon_i$.*

□

**Lecture 8** **May 27, 2019**

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**Definition 8.1**

*A square hermitian matrix $A$ of size $n$.*

field of fractions