Contents

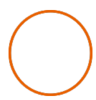
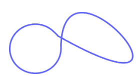
Lecture 1 Basic definitions — February 25, 2019

Definition 1.1
A knot K in S^3 is a smooth (PL - smooth) embedding of a circle S^1 in S^3:

$$\varphi : S^1 \hookrightarrow S^3$$

Usually we think about a knot as an image of an embedding: $K = \varphi(S^1)$.

Example 1.1

- *Knots:*  *(unknot),*  *(trefoil).*
- *Not knots:*  *(it is not an injection),*  *(it is not smooth).*

Definition 1.2
Two knots $K_0 = \varphi_0(S^1)$, $K_1 = \varphi_1(S^1)$ are equivalent if the embeddings φ_0 and φ_1 are isotopic, that is there exists a continues function

$$\Phi : S^1 \times [0,1] \hookrightarrow S^3$$
$$\Phi(x,t) = \Phi_t(x)$$

such that Φ_t is an embedding for any $t \in [0,1]$, $\Phi_0 = \varphi_0$ and $\Phi_1 = \varphi_1$.

Theorem 1.1
Two knots K_0 and K_1 are isotopic if and only if they are ambient isotopic, i.e. there exists a family of self-diffeomorphisms $\Psi = \{\psi_t : t \in [0,1]\}$ such that:

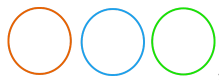
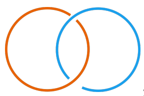
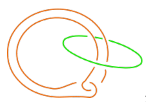
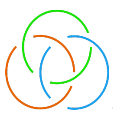
$$\psi(t) = \psi_t \text{ is continius on } t \in [0,1]$$
$$\psi_t : S^3 \hookrightarrow S^3,$$
$$\psi_0 = id,$$
$$\psi_1(K_0) = K_1.$$

Definition 1.3
A knot is trivial (unknot) if it is equivalent to an embedding $\varphi(t) = (\cos t, \sin t, 0)$, where $t \in [0, 2\pi]$ is a parametrisation of S^1.

Definition 1.4

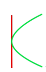
A link with k - components is a (smooth) embedding of $\overbrace{S^1 \sqcup \ldots \sqcup S^1}^{k}$ in S^3

Example 1.2
Links:

- *a trivial link with 3 components:* ,
- *a hopf link:* ,
- *a Whitehead link:* ,
- *Borromean link:* .

Definition 1.5
A link diagram D_π is a picture over projection π of a link L in $\mathbb{R}^3$ (S^3) to $\mathbb{R}^2$ (S^2) such that:

(1) $D_{\pi|_L}$ is non degenerate: ,

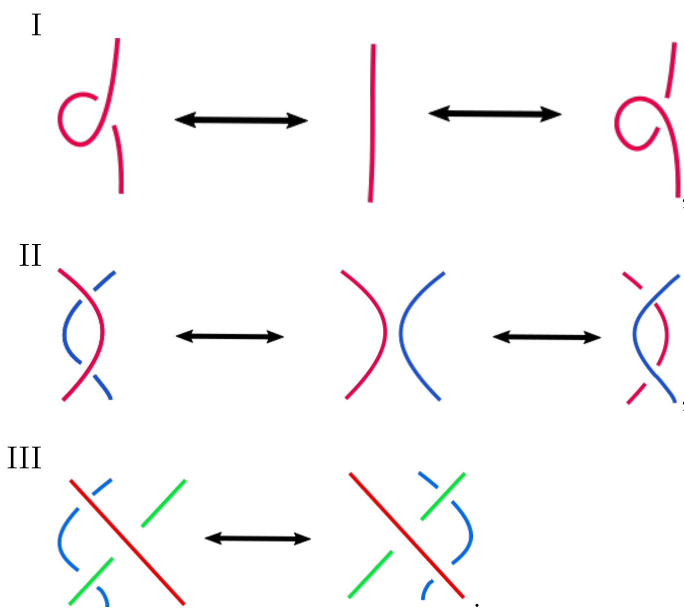
(2) the double points are not degenerate: ,

(3) there are no triple point: .

There are under- and overcrossings (tunnels and bridges) on a link diagrams with an obvious meaning.
Every link admits a link diagram.
Let D be a diagram of an oriented link (to each component of a link we add an arrow in the diagram).
We can distinguish two types of crossings: right-handed (), called a positive crossing, and left-handed (), called a negative crossing.

Reidemeister moves

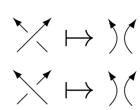
A Reidemeister move is one of the three types of operation on a link diagram as shown below:



Theorem 1.2 (Reidemeister, 1927)
Two diagrams of the same link can be deformed into each other by a finite sequence of Reidemeister moves (and isotopy of the plane).

Seifert surface

Let D be an oriented diagram of a link L. We change the diagram by smoothing each crossing:



We smooth all the crossings, so we get a disjoint union of circles on the plane. Each circle bounds a disks in $\mathbb{R}^3$ (we choose disks that don't intersect). For each smoothed crossing we add a twisted band: right-handed for a positive and left-handed for a negative one. We get an orientable surface Σ such that $\partial\Sigma = L$.

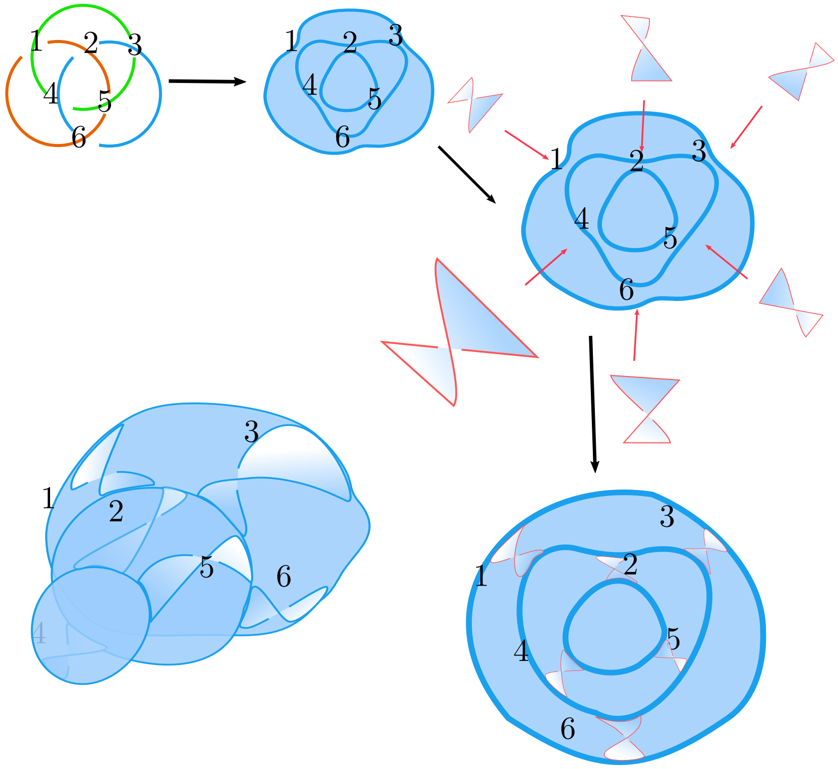


Figure 1: Constructing a Seifert surface.

Note: the obtained surface isn't unique and in general doesn't need to be connected, but by taking connected sum of all components we can easily get a connected surface (i.e. we take two disconnected components and cut a disk in each of them: D_1 and D_2; now we glue both components on the boundaries: ∂D_1 and ∂D_2.

Theorem 1.3 (Seifert)
Every link in S^3 bounds a surface Σ that is compact, connected and orientable. Such a surface is called a Seifert surface.

Definition 1.6
The three genus $g_3(K)$ ($g(K)$) of a knot K is the minimal genus of a Seifert surface Σ for K.

Corollary 1.1
A knot K is trivial if and only $g_3(K) = 0$.

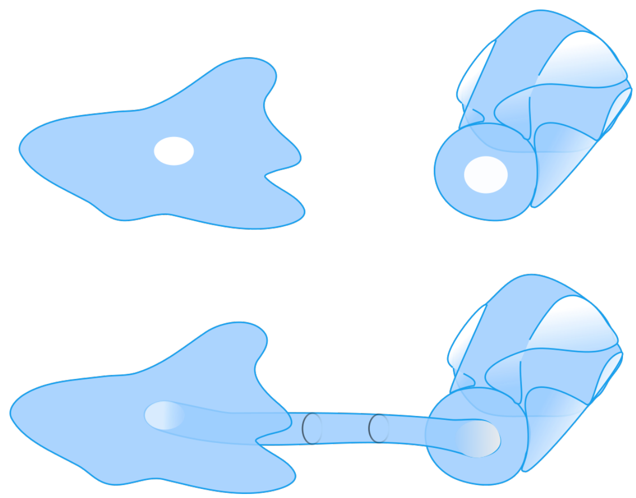


Figure 2: Connecting two surfaces.

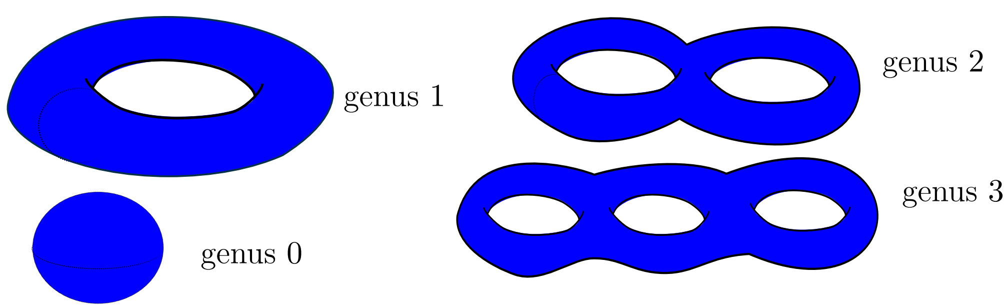


Figure 3: Genus of an orientable surface.

Remark: there are knots that admit non isotopic Seifert surfaces of minimal genus (András Juhász, 2008).

Definition 1.7
Suppose α and β are two simple closed curves in $\mathbb{R}^3$. On a diagram L consider all crossings between α and β. Let N_+ be the number of positive crossings, N_- - negative. Then the linking number: $\operatorname{lk}(\alpha, \beta) = \frac{1}{2}(N_+ - N_-)$.

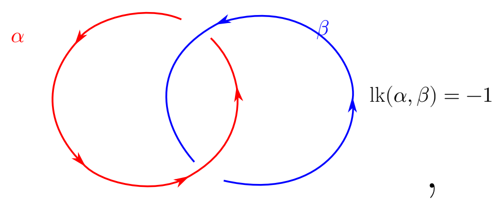
Definition 1.8
Let α and β be two disjoint simple cross curves in S^3. Let $\nu(\beta)$ be a tubular neighbourhood of β. The linking number can be interpreted via first homology

group, where $\operatorname{lk}(\alpha, \beta)$ *is equal to evaluation of* α *as element of first homology group of the complement of* β*:*

$$\alpha \in H_1(S^3 \setminus \nu(\beta), \mathbb{Z}) \cong \mathbb{Z}.$$

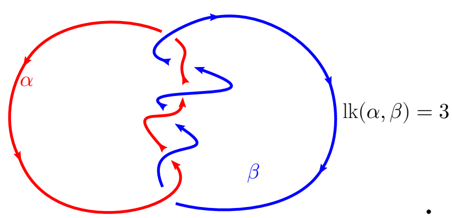
Example 1.3

- *Hopf link:*



,

- $T(6,2)$ *link:*



.

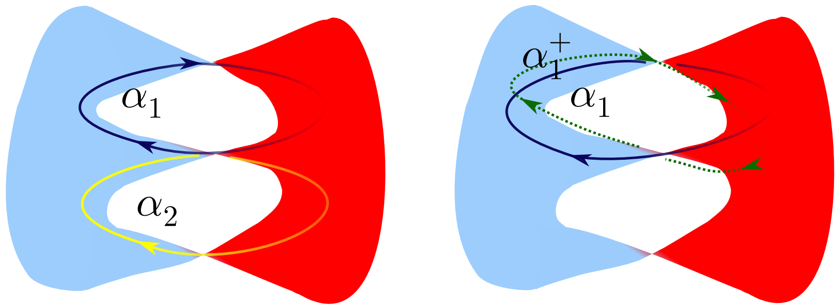
Fact 1.1

$$g_3(\Sigma) = \frac{1}{2} b_1(\Sigma) = \frac{1}{2} \dim_{\mathbb{R}} H_1(\Sigma, \mathbb{R}),$$

where b_1 *is first Betti number of* Σ*.*

Seifert matrix

Let L be a link and Σ be an oriented Seifert surface for L. Choose a basis for $H_1(\Sigma, \mathbb{Z})$ consisting of simple closed $\alpha_1, \ldots, \alpha_n$. Let $\alpha_1^+, \ldots \alpha_n^+$ be copies of α_i lifted up off the surface (push up along a vector field normal to Σ). Note that elements α_i are contained in the Seifert surface while all α_i^+ are don't intersect the surface. Let $\operatorname{lk}(\alpha_i, \alpha_j^+) = \{a_{ij}\}$. Then the matrix $S = \{a_{ij}\}_{i,j=1}^n$ is called a Seifert matrix for L. Note that by choosing a different basis we get a different matrix.



Theorem 1.4
The Seifert matrices S_1 and S_2 for the same link L are S-equivalent, that is, S_2 can be obtained from S_1 by a sequence of following moves:

(1) $V \to AVA^T$, *where A is a matrix with integer coefficients,*

(2) $$V \to \left(\begin{array}{ccc|cc} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right) \quad \textit{or} \quad V \to \left(\begin{array}{ccc|cc} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right)$$

(3) inverse of (2)

Lecture 2 Alexander polynomial March 4, 2019

Existence of Seifert surface - second proof

Proof. (Theorem 1.3)
Let $K \in S^3$ be a knot and $N = \nu(K)$ be its tubular neighbourhood. Because K and N are homotopy equivalent, we get:

$$H^1(S^3 \setminus N) \cong H^1(S^3 \setminus K).$$

Let us consider a long exact sequence of cohomology of a pair $(S^3, S^3 \setminus N)$ with integer coefficients:

$$
\begin{array}{cccc}
 & \mathbb{Z} & & \\
 & \wr\| & & \\
 & H^0(S^3) \to & H^0(S^3 \setminus N) \to & \\
 & & & \\
\to H^1(S^3, S^3 \setminus N) \to & H^1(S^3) \to & H^1(S^3 \setminus N) \to & \\
 & \wr\| & & \\
 & 0 & & \\
 & \wr\| & & \\
\to H^2(S^3, S^3 \setminus N) \to & H^2(S^3) \to & H^2(S^3 \setminus N) \to & \\
 & & & \\
\to H^3(S^3, S^3 \setminus N) \to & H^3(S) \to & 0 & \\
 & \wr\| & & \\
 & \mathbb{Z} & &
\end{array}
$$

$$
\begin{aligned}
N &\cong D^2 \times S^1 \\
\partial N &\cong S^1 \times S^1 \\
H^1(N, \partial N) &\cong \mathbb{Z} \oplus \mathbb{Z}
\end{aligned}
$$

$$H^*(S^3, S^3 \setminus N) \cong H^*(N, \partial N)$$

$$H^1(S^3 \setminus N) \cong H^1(S^3 \setminus K) \cong \mathbb{Z}$$

$$
\begin{array}{ccc}
H^1(S^3 \setminus K) & \longrightarrow & H^1(N \setminus K) \\
\Big\downarrow \widetilde{\Theta} & & \Big\downarrow \Theta \\
[S^3 \setminus K, S^1] & \longrightarrow & [N \setminus K, S^1]
\end{array}
$$

$\Sigma = \widetilde{\Theta}^{-1}(X)$ is a surface, such that $\partial\Sigma = K$, so it is a Seifert surface. □

Alexander polynomial

Definition 2.1
Let S be a Seifert matrix for a knot K. The Alexander polynomial $\Delta_K(t)$ is a Laurent polynomial:

$$\Delta_K(t) := \det(tS - S^T) \in \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$$

Theorem 2.1
$\Delta_K(t)$ is well defined up to multiplication by $\pm t^k$, for $k \in \mathbb{Z}$.

Proof. We need to show that $\Delta_K(t)$ doesn't depend on S-equivalence relation.

(1) Suppose $S' = CSC^T$, $C \in \mathrm{GL}(n, \mathbb{Z})$ (matrices invertible over $\mathbb{Z}$). Then $\det C = 1$ and:

$$\det(tS' - S'^T) = \det(tCSC^T - (CSC^T)^T) =$$
$$\det(tCSC^T - CS^TC^T) = \det C(tS - S^T)C^T = \det(tS - S^T)$$

(2) Let

$$A := t\left(\begin{array}{ccc|cc} & & & * & 0 \\ & S & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right) - \left(\begin{array}{ccc|cc} & & & * & 0 \\ & S^T & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} & & & * & 0 \\ & tS - S^T & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \dots & * & 0 & -1 \\ 0 & \dots & 0 & t & 0 \end{array}\right)$$

Using the Laplace expansion we get $\det A = \pm t \det(tS - S^T)$.

□

Example 2.1
If K is a trefoil then we can take $S = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Then

$$\Delta_K(t) = \det \begin{pmatrix} -t+1 & -t \\ 1 & -t+1 \end{pmatrix} = (t-1)^2 + t = t^2 - t + 1 \neq 1 \Rightarrow \textit{trefoil is not trivial.}$$

Fact 2.1
$\Delta_K(t)$ is symmetric.

Proof. Let S be an $n \times n$ matrix.

$$\Delta_K(t^{-1}) = \det(t^{-1}S - S^T) = (-t)^{-n} \det(tS^T - S) =$$
$$(-t)^{-n} \det(tS - S^T) = (-t)^{-n} \Delta_K(t)$$

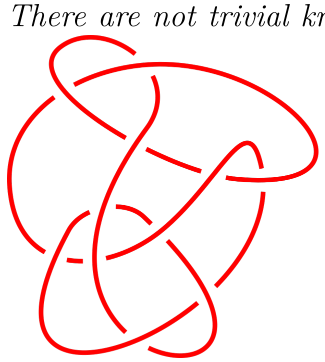
If K is a knot, then n is necessarily even, and so $\Delta_K(t^{-1}) = t^{-n}\Delta_K(t)$. □

Lemma 2.1

$$\frac{1}{2}\deg \Delta_K(t) \leq g_3(K), \;\; \textit{where } deg(a_n t^n + \dots + a_1 t^l) = k - l.$$

Proof. If Σ is a genus g - Seifert surface for K then $H_1(\Sigma) = \mathbb{Z}^{2g}$, so S is an $2g \times 2g$ matrix. Therefore $\det(tS - S^T)$ is a polynomial of degree at most $2g$. □

Example 2.2
There are not trivial knots with Alexander polynomial equal 1*, for example:*



$\Delta_{11n34} \equiv 1.$

Decomposition of 3-sphere

We know that 3 - sphere can be obtained by gluing two solid tori: $S^3 = \partial D^4 = \partial(D^2 \times D^2) = (D^2 \times S^1) \cup (S^1 \times D^2)$. So the complement of solid torus in S^3 is another solid torus.
Analytically it can be describes as follow. Take $(z_1, z_2) \in \mathbb{C}$ such that $max(|z_1|, |z_2|) = 1$. Define following sets: $S_1 = \{(z_1, z_2) \in S^3 : |z_1| = 0\} \cong S^1 \times D^2$ and $S_2 = \{(z_1, z_2) \in S^3 : |z_2| = 1\} \cong D^2 \times S^1$. The intersection $S_1 \cap S_2 = \{(z_1, z_2) : |z_1| = |z_2| = 1\} \cong S^1 \times S^1$

Dehn lemma and sphere theorem

Lemma 2.2 (Dehn)
Let M *be a* 3*-manifold and* $D^2 \xrightarrow{f} M^3$ *be a map of a disk such that* $f|_{\partial D^2}$ *is an embedding. Then there exists an embedding* $D^2 \xhookrightarrow{g} M$ *such that:*

$$g|_{\partial D^2} = f|_{\partial D^2}.$$

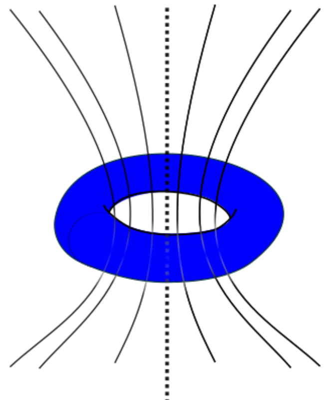


Figure 4: The complement of solid torus in S^3 is another solid torus.

Remark: Dehn lemma doesn't hold for dimension four.
Let M be connected, compact three manifold with boundary. Suppose $\pi_1(\partial M) \longrightarrow \pi_1(M)$ has non-trivial kernel. Then there exists a map $f : (D^2, \partial D^2) \longrightarrow (M, \partial M)$ such that $f\big|_{\partial D^2}$ is non-trivial loop in ∂M.

Theorem 2.2 (Sphere theorem)
Suppose $\pi_1(M) \neq 0$. Then there exists an embedding $f : S^2 \hookrightarrow M$ that is homotopy non-trivial.

Problem 2.1
Prove that S^3 K is Eilenberg–MacLane space of type $K(\pi, 1)$.

Corollary 2.1
Suppose $K \subset S^3$ and $\pi_1(S^3 \setminus K)$ is infinite cyclic ($\mathbb{Z}$). Then K is trivial.

Proof. Let N be a tubular neighbourhood of a knot K and $M = S^3 \setminus N$ its complement. Then $\partial M = S^1 \times S^1$. Let $f : \pi_1(\partial M) \longrightarrow \pi_1(M)$. If $\pi_1(M)$ is infinite cyclic group then the map f is non-trivial. Suppose $\lambda \in \ker(\pi_1(S^1 \times S^1) \longrightarrow \pi_1(M)$. There is a map $g : (D^2, \partial D^2) \longrightarrow (M, \partial M)$ such that $g(\partial D^2) = \lambda$. By Dehn's lemma there exists an embedding $h : (D^2, \partial D^2) \hookrightarrow (M, \partial M$ such that $h\big|_{\partial D^2} = f\big|_{\partial D^2}$ and $h(\partial D^2) = \lambda$. Let Σ be a union of the annulus and the image of ∂D^2.
???? g_3?
If $g(\Sigma) = 0$, then K is trivial.
Now we should proof that:

$$H_1(M) \cong \mathbb{Z} \Longrightarrow \lambda \in \ker(\pi_1(S^1 \times S^1) \longrightarrow \pi_1(M)).$$

Choose a meridian μ such that $\text{lk}(\mu, K) = 1$. Recall the definition of linking

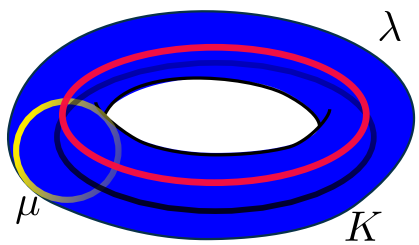


Figure 5: μ is a meridian and λ is a longitude.

number via homology group (Definition 1.8). $[\mu]$ represents the generator of $H_1(S^3 \setminus K, \mathbb{X})$. From definition of λ we know that λ is trivial in $H_1(M)$ ($\text{lk}(\lambda, K) = 0$, therefore $[\lambda]$ was trivial in $pi_1(M)$). If K is non-trivial then λ is non-trivial in $\pi_1(M)$, but it is trivial in $H_1(M)$. □

Lecture 3 March 11, 2019

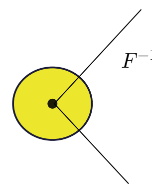
Example 3.1

$$F : \mathbb{C}^2 \to \mathbb{C} \text{ a polynomial}$$
$$F(0) = 0$$

????????????
as a corollary we see that $K_T^{n,}$????
is not slice unless $m = 0$.

Theorem 3.1
The map $j : \mathcal{C} \longrightarrow \mathbb{Z}^\infty$ is a surjection that maps K_n to a linear independent set. Moreover $\mathcal{C} \cong \mathbb{Z}$

Fact 3.1 (Milnor Singular Points of Complex Hypersurfaces)



An oriented knot is called negative amphichiral if the mirror image $m(K)$ of K is equivalent the reverse knot of K: K^r.

Problem 3.1
Prove that if K is negative amphichiral, then $K \# K = 0$ in $\mathcal{C}$.

Example 3.2
Figure 8 knot is negative amphichiral.

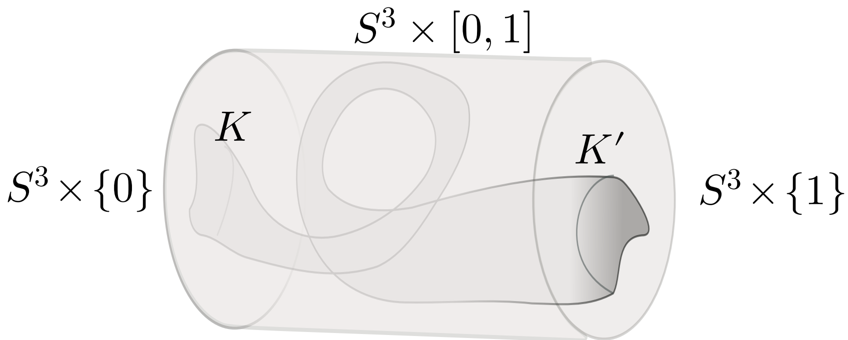
Definition 3.1
A link L is fibered if there exists a map $\phi : S^3 \setminus L \longleftarrow S^1$ which is locally trivial fibration.

Lecture 4 Concordance group March 18, 2019

Definition 4.1
Two knots K and K' are called (smoothly) concordant if there exists an annulus A that is smoothly embedded in $S^3 \times [0,1]$ such that

$$\partial A = K' \times \{1\} \sqcup K \times \{0\}.$$

Definition 4.2
A knot K is called (smoothly) slice if K is smoothly concordant to an unknot. Put differently: a knot K is smoothly slice if and only if K bounds a smoothly embedded disk in B^4.



Let $m(K)$ denote a mirror image of a knot K.

Fact 4.1
For any K, $K\#m(K)$ is slice.

Fact 4.2
Concordance is an equivalence relation.

Fact 4.3
If $K_1 \sim {K_1}'$ and $K_2 \sim {K_2}'$, then $K_1\#K_2 \sim {K_1}'\#{K_2}'$.

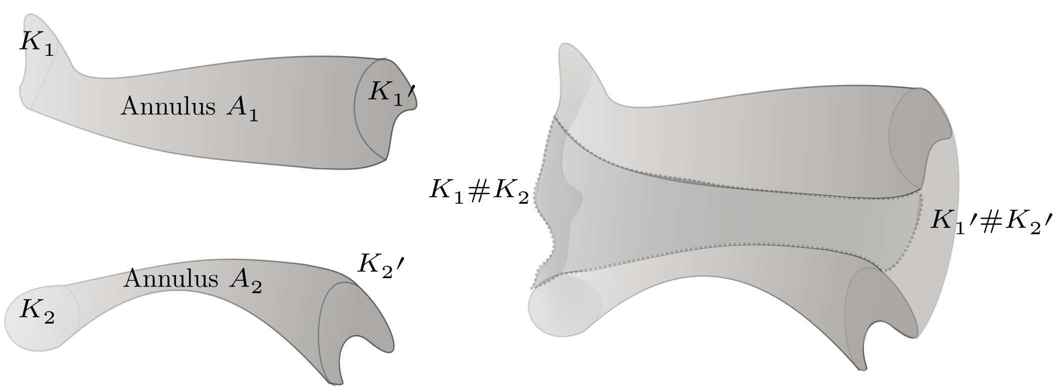


Figure 6: Sketch for Fact 4.3.

Fact 4.4
$K\#m(K) \sim$ the unknot.

Theorem 4.1
Let $\mathcal{C}$ denote a set of all equivalent classes for knots and $\{0\}$ denote class of all knots concordant to a trivial knot. $\mathcal{C}$ is a group under taking connected sums. The neutral element in the group is $\{0\}$ and the inverse element of an element $\{K\} \in \mathcal{C}$ is $-\{K\} = \{mK\}$.

Fact 4.5
The figure eight knot is a torsion element in $\mathcal{C}$ ($2K \sim$ the unknot).

Problem 4.1 (open)
Are there in concordance group torsion elements that are not 2 torsion elements?

Remark: $K \sim K' \Leftrightarrow K \# -K'$ is slice.

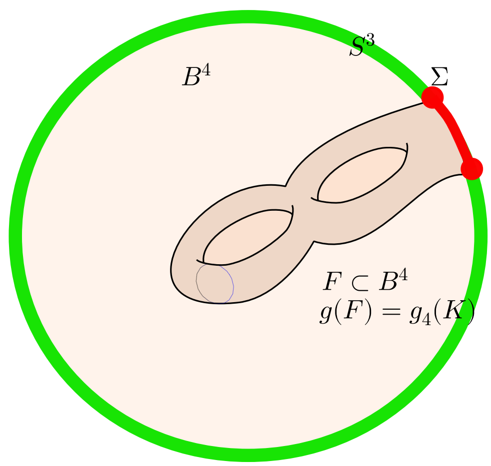


Figure 7: $Y = F \cup \Sigma$ is a smooth close surface.

Pontryagin-Thom construction tells us that there exists a compact three - manifold $\Omega \subset B^4$ such that $\partial\Omega = Y$. Suppose Σ is a Seifert surface and V a Seifert form defined on Σ: $(\alpha, \beta) \mapsto \mathrm{lk}(\alpha, \beta^+)$. Suppose $\alpha, \beta \in H_1(\Sigma, \mathbb{Z})$, i.e. there are cycles and $\alpha, \beta \in \ker(H_1(\Sigma, \mathbb{Z}) \longrightarrow H_1(\Omega, \mathbb{Z}))$. Then there are two cycles $A, B \in \Omega$ such that $\partial A = \alpha$ and $\partial B = \beta$. Let B^+ be a push off of B in the positive normal direction such that $\partial B^+ = \beta^+$. Then $\mathrm{lk}(\alpha, \beta^+) = A \cdot B^+$. But A and B are disjoint, so $\mathrm{lk}(\alpha, \beta^+) = 0$. Then the Seifert form is zero. ????????????????

Let us consider following maps:

$$\Sigma \stackrel{\phi}{\hookrightarrow} Y \stackrel{\psi}{\hookrightarrow} \Omega.$$

Let ϕ_* and ψ_* be induced maps on the homology group. If an element $\gamma \in \ker(H_1(\Sigma, \mathbb{Z}) \longrightarrow H_1(\Omega, \mathbb{Z}))$, then $\gamma \in \ker \phi_*$ or $\gamma \in \ker \psi_*$.
????????????

Proposition 4.1

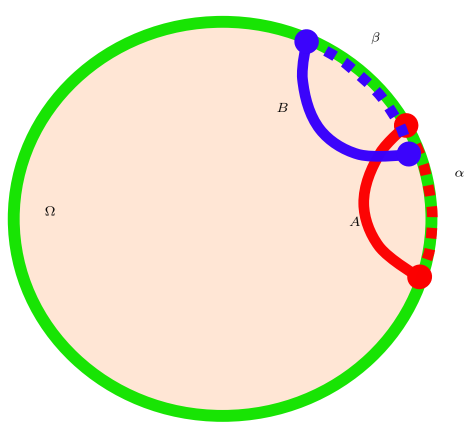
$$\dim \ker(H_1(Y, \mathbb{Z}) \longrightarrow H_1(\Omega, \mathbb{Z})) = \frac{1}{2} b_1(Y),$$

where b_1 *is first Betti number.*

Proof.

$$\begin{aligned} 0 \to H_3(\Omega) \to H_3(\Omega, Y) \to \\ \to H_2(Y) \to H_2(\Omega) \to H_2(\Omega, Y) \to \\ \to H_1(Y) \to 1(\Omega) \to H_1(\Omega, Y) \to \\ \to H_0(Y) \to H_0(\Omega) \to 0 \end{aligned}$$

□



Lecture 5 — March 25, 2019

Definition 5.1
The (smooth) four genus $g_4(K)$ is the minimal genus of the surface $\Sigma \in B^4$ such that Σ is compact, orientable and $\partial\Sigma = K$.

Remark: 3 - genus is additive under taking connected sum, but 4 - genus is not.

Lecture 6 — April 8, 2019

X is a closed orientable four-manifold. Assume $\pi_1(X) = 0$ (it is not needed to define the intersection form). In particular $H_1(X) = 0$. H_2 is free (exercise).

$$H_2(X, \mathbb{Z}) \xrightarrow{\text{Poincaré duality}} H^2(X, \mathbb{Z}) \xrightarrow{\text{evaluation}} \operatorname{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$$

Intersection form: $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ - symmetric, non singular.
Let A and B be closed, oriented surfaces in X.

Proposition 6.1
$A \cdot B$ doesn't depend of choice of A and B in their homology classes.

Proof. By Poincaré duality we know that:

$$\begin{aligned} H_3(\Omega, Y) &\cong H^0(\Omega), \\ H_2(Y) &\cong H^0(Y), \\ H_2(\Omega) &\cong H^1(\Omega, Y), \\ H_2(\Omega, Y) &\cong H^1(\Omega). \end{aligned}$$

Therefore $\dim_{\mathbb{Q}} {}^{H_1(Y)}\!/_{V} = \dim_{\mathbb{Q}} V$.
Suppose $g(K) = 0$ (K is slice). Then $H_1(\Sigma, \mathbb{Z}) \cong H_1(Y, \mathbb{Z})$. Let g_Σ be the genus of Σ, $\dim H_1(Y, \mathbb{Z}) = 2g_\Sigma$. Then the Seifert form V on a K has a subspace of dimension g_Σ on which it is zero:

$$V = g_\Sigma \left\{ \overbrace{\begin{pmatrix} 0 & ... & 0 & * & ... & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & ... & 0 & * & ... & * \\ * & ... & * & * & ... & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & ... & * & * & ... & * \end{pmatrix}}^{g_\Sigma} \right._{2g_\Sigma \times 2g_\Sigma}$$

□

Lecture 7 **April 15, 2019**

In other words:
Choose a basis $(b_1, ..., b_i)$
???
of $H_2(Y, \mathbb{Z}$, then $A = (b_i, b_y)$
??
is a matrix of intersection form:

$$\mathbb{Z}^n / A\mathbb{Z}^n \cong H_1(Y, \mathbb{Z}).$$

In particular $\mid \det A \mid = \# H_1(Y, \mathbb{Z})$.
That means - what is happening on boundary is a measure of degeneracy.

$$\begin{array}{ccccc} H_1(Y,\mathbb{Z}) & \times & H_1(Y,\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \text{ - a linking form} \\ \wr\| & & \wr\| & & \\ \mathbb{Z}^n/A\mathbb{Z} & & \mathbb{Z}^n/A\mathbb{Z} & & \end{array}$$

$$(a, b) \mapsto aA^{-1}b^T$$

????????????????????????????????

The intersection form on a four-manifold determines the linking on the boundary.

Let $K \in S^1$ be a knot, $\Sigma(K)$ its double branched cover. If V is a Seifert matrix for K, then $H_1(\Sigma(K), \mathbb{Z}) \cong {\mathbb{Z}^n}/{A\mathbb{Z}}$ where $A = V \times V^T$, $n = \operatorname{rank} V$. Let X be the four-manifold obtained via the double branched cover of B^4

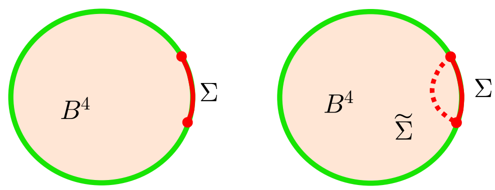


Figure 8: Pushing the Seifert surface in 4-ball.

branched along $\widetilde{\Sigma}$.

Fact 7.1

- *X is a smooth four-manifold,*
- *$H_1(X, \mathbb{Z}) = 0$,*
- *$H_2(X, \mathbb{Z}) \cong \mathbb{Z}^n$*
- *The intersection form on X is $V + V^T$.*

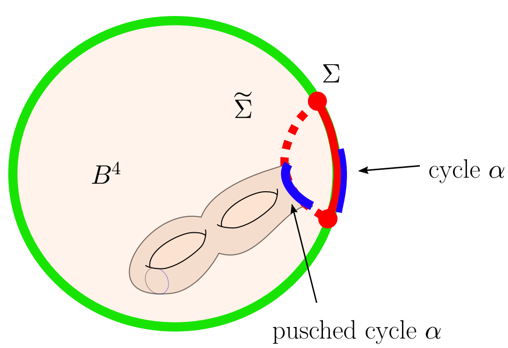


Figure 9: Cycle pushed in 4-ball.

Let $Y = \Sigma(K)$. Then:

$$H_1(Y, \mathbb{Z}) \times H_1(Y, \mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$
$$(a, b) \mapsto aA^{-1}b^T, \qquad A = V + V^T.$$

???????????????????????????

$$H_1(Y, \mathbb{Z}) \cong \mathbb{Z}^n / A\mathbb{Z}$$
$$A \longrightarrow BAC^T \quad \text{Smith normal form}$$

??????????????????????
In general

Lecture 8 May 20, 2019

Let M be compact, oriented, connected four-dimensional manifold. If $H_1(M, \mathbb{Z}) = 0$ then there exists a bilinear form - the intersection form on M:

$$\begin{array}{ccccc} H_2(M, \mathbb{Z}) & \times & H_2(M, \mathbb{Z}) \longrightarrow & \mathbb{Z} \\ \wr\Vert & & & \\ \mathbb{Z}^n & & & \end{array}$$

Let us consider a specific case: M has a boundary $Y = \partial M$. Betti number $b_1(Y) = 0$, $H_1(Y, \mathbb{Z})$ is finite. Then the intersection form can be degenerated in the sense that:

$$H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \longrightarrow \mathbb{Z} \qquad H_2(M, \mathbb{Z}) \longrightarrow \operatorname{Hom}(H_2(M, \mathbb{Z}), \mathbb{Z})$$
$$(a, b) \mapsto \mathbb{Z} \qquad a \mapsto (a, \_)H_2(M, \mathbb{Z})$$

has coker precisely $H_1(Y, \mathbb{Z})$.
???????????????
Let $K \subset S^3$ be a knot,

$X = S^3 \setminus K$ - a knot complement,
$\widetilde{X} \xrightarrow{\rho} X$ - an infinite cyclic cover (universal abelian cover).

$$\pi_1(X) \longrightarrow {}^{\pi_1(X)}\big/_{[\pi_1(X), \pi_1(X)]} = H_1(X, \mathbb{Z}) \cong \mathbb{Z}$$

$C_*(\widetilde{X})$ has a structure of a $\mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$ module.
$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}])$ - Alexander module,

$$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \times H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \longrightarrow {}^{\mathbb{Q}}\big/_{\mathbb{Z}[t, t^{-1}]}$$

Fact 8.1

$$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \cong {}^{\mathbb{Z}[t, t^{-1}]^n}\big/_{(tV - V^T)\mathbb{Z}[t, t^{-1}]^n},$$

where V is a Seifert matrix.

Fact 8.2

$$H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \times H_1(\widetilde{X}, \mathbb{Z}[t, t^{-1}]) \longrightarrow {}^{\mathbb{Q}}\big/_{\mathbb{Z}[t, t^{-1}]}$$
$$(\alpha, \beta) \mapsto \alpha^{-1}(t-1)(tV - V^T)^{-1}\beta$$

Note that $\mathbb{Z}$ is not PID. Therefore we don't have primer decomposition of this moduli. We can simplify this problem by replacing $\mathbb{Z}$ by $\mathbb{R}$. We lose some date by doing this transition.

$\xi \in S^1 \setminus \{\pm 1\} \quad p_\xi = (t - \xi)(t - \xi^{-1})t^{-1}$
$\xi \in \mathbb{R} \setminus \{\pm 1\} \quad q_\xi = (t - \xi)(t - \xi^{-1})t^{-1}$
$\xi \notin \mathbb{R} \cup S^1 \quad q_\xi = (t - \xi)(t - \bar{\xi})(t - \xi^{-1})(t - \bar{\xi}^{-1})t^{-2}$
$\Lambda = \mathbb{R}[t, t^{-1}]$
Then: $H_1(\widetilde{X}, \Lambda) \cong \bigoplus_{\substack{\xi \in S^1 \setminus \{\pm 1\} \\ k \geq 0}} ({}^{\Lambda}\big/_{p_\xi^k})^{n_k, \xi} \oplus \bigoplus_{\substack{\xi \notin S^1 \\ l \geq 0}} ({}^{\Lambda}\big/_{q_\xi^l})^{n_l, \xi}$

We can make this composition orthogonal with respect to the Blanchfield paring.

Historical remark:

- John Milnor, *On isometries of inner product spaces*, 1969,
- Walter Neumann, *Invariants of plane curve singularities* , 1983,
- András Némethi, *The real Seifert form and the spectral pairs of isolated hypersurfaceenumerate singularities*, 1995,
- Maciej Borodzik, Stefan Friedl *The unknotting number and classical invariants II*, 2014.

Let $p = p_\xi$, $k \geq 0$.

$$\Lambda/_{p^k\Lambda} \times \Lambda/_{p^k\Lambda} \longrightarrow \mathbb{Q}(t)/_{\Lambda}$$
$$(1,1) \mapsto \kappa$$
$$\text{Now: } (p^k \cdot 1, 1) \mapsto 0$$
$$p^k\kappa = 0 \in \mathbb{Q}(t)/_{\Lambda}$$
$$\text{therfore } p^k\kappa \in \Lambda$$
$$\text{we have } (1,1) \mapsto \frac{h}{p^k}$$

h is not uniquely defined: $h \to h + gp^k$ doesn't affect paring.
Let $h = p^k\kappa$.

Example 8.1

$$\phi_0((1,1)) = \frac{+1}{p}$$
$$\phi_1((1,1)) = \frac{-1}{p}$$

ϕ_0 and ϕ_1 are not isomorphic.

Proof. Let $\Phi : \Lambda/_{p^k\Lambda} \longrightarrow \Lambda/_{p^k\Lambda}$ be an isomorphism.
Let: $\Phi(1) = g \in \lambda$

$$\Lambda/_{p^k\Lambda} \xrightarrow{\Phi} \Lambda/_{p^k\Lambda}$$
$$\phi_0((1,1)) = \frac{1}{p^k} \qquad \phi_1((g,g)) = \frac{1}{p^k} \quad (\Phi \text{ is an isometry}).$$

Suppose for the paring $\phi_1((g,g)) = \frac{1}{p^k}$ we have $\phi_1((1,1)) = \frac{-1}{p^k}$. Then:

$$\begin{aligned}
\frac{-g\bar{g}}{p^k} &= \frac{1}{p^k} \in \mathbb{Q}(t)\big/_{\Lambda} \\
\frac{-g\bar{g}}{p^k} &- \frac{1}{p^k} \in \Lambda \\
-g\bar{g} &\equiv 1 \pmod{p} \text{ in } \Lambda \\
-g\bar{g} - 1 &= p^k\omega \text{ for some } \omega \in \Lambda
\end{aligned}$$

evalueting at ξ:

$$\overbrace{-g(\xi)g(\xi^{-1})}^{>0} - 1 = 0 \quad \Rightarrow\Leftarrow$$

□

???????????????????

$$\begin{aligned}
g &= \sum g_i t^i \\
\bar{g} &= \sum g_i t^{-i} \\
\bar{g}(\xi) &= \sum g_i \xi^i \quad \xi \in S^1 \\
\bar{g}(\xi) &= g(\bar{\xi})
\end{aligned}$$

Suppose $g = (t-\xi)^\alpha g'$. Then $(t-\xi)^{k-\alpha}$ goes to 0 in $\Lambda\big/_{p^k\Lambda}$.

Theorem 8.1
Every sesquilinear non-degenerate pairing

$$\Lambda\big/_{p^k} \times \Lambda\big/_{p} \longleftrightarrow \frac{h}{p^k}$$

is isomorphic either to the pairing wit $h = 1$ or to the paring with $h = -1$ depending on sign of $h(\xi)$ (which is a real number).

Proof. There are two steps of the proof:

1. Reduce to the case when h has a constant sign on S^1.
2. Prove in the case, when h has a constant sign on S^1.

Lemma 8.1
If P is a symmetric polynomial such that $P(\eta) \geq 0$ for all $\eta \in S^1$, then P can be written as a product $P = g\bar{g}$ for some polynomial g.

Sketch of proof. Induction over $\deg P$.
Let $\zeta \notin S^1$ be a root of P, $P \in \mathbb{R}[t, t^{-1}]$. Assume $\zeta \notin \mathbb{R}$. We know that polynomial P is divisible by $(t-\zeta)$, $(t-\bar{\zeta})$, $(t^{-1}-\zeta)$ and $(t^{-1}-\bar{\zeta})$. Therefore:

$$P' = \frac{P}{(t-\zeta)(t-\bar{\zeta})(t^{-1}-\zeta)(t^{-1}-\bar{\zeta})}$$
$$P' = g'\bar{g}$$

We set $g = g'(t-\zeta)(t-\bar{\zeta})$ and $P = g\bar{g}$. Suppose $\zeta \in S^1$. Then $(t-\zeta)^2 \mid P$ (at least - otherwise it would change sign). Therefore:

$$P' = \frac{P}{(t-\zeta)^2(t^{-1}-\zeta)^2}$$
$$g = (t-\zeta)(t^{-1}-\zeta)g' \quad \text{etc.}$$

The map $(1,1) \mapsto \frac{h}{p^k} = \frac{g\bar{g}h}{p^k}$ is isometric whenever g is coprime with P. □

Lemma 8.2
Suppose A and B are two symmetric polynomials that are coprime and that $\forall z \in S^1$ either $A(z) > 0$ or $B(z) > 0$. Then there exist symmetric polynomials P, Q such that $P(z), Q(z) > 0$ for $z \in S^1$ and $PA + QB \equiv 1$.

Idea of proof. For any z find an interval (a_z, b_z) such that if $P(z) \in (a_z, b_z)$ and $P(z)A(z) + Q(z)B(z) = 1$, then $Q(z) > 0$, $x(z) = \frac{az+bz}{i}$ is a continues function on S^1 approximating z by a polynomial .
?????????????????????????

$$(1,1) \mapsto \frac{h}{p^k} \mapsto \frac{g\bar{g}h}{p^k}$$
$$g\bar{g}h + p^k\omega = 1$$

Apply Lemma 8.2 for $A = h$, $B = p^{2k}$. Then, if the assumptions are satisfied,

$$\begin{aligned} Ph + Qp^{2k} &= 1 \\ p > 0 \Rightarrow p &= g\bar{g} \\ p &= (t-\xi)(t-\bar{\xi})t^{-1} \\ \text{so } p &\geq 0 \text{ on } S^1 \\ p(t) = 0 \Leftrightarrow t &= \xi or t = \bar{\xi} \\ h(\xi) &> 0 \\ h(\bar{\xi}) &> 0 \\ g\bar{g}h + Qp^{2k} &= 1 \\ g\bar{g}h &\equiv 1 \mod p^{2k} \\ g\bar{g} &\equiv 1 \mod p^k \end{aligned}$$

??????????????????????????????

If P has no roots on S^1 then $B(z) > 0$ for all z, so the assumptions of Lemma 8.2 are satisfied no matter what A is. □

????????????????

$$\begin{aligned} (\Lambda/_{p_\xi^k} \times \Lambda/_{p_\xi^k}) &\longrightarrow \frac{\epsilon}{p_\xi^k}, \quad \xi \in S^1 \setminus \{\pm 1\} \\ (\Lambda/_{q_\xi^k} \times \Lambda/_{q_\xi^k}) &\longrightarrow \frac{1}{q_\xi^k}, \quad \xi \notin S^1 \end{aligned}$$

?????????????????? 1 ?? epsilon?

Theorem 8.2

(Matumoto, Borodzik-Conway-Politarczyk) Let K be a knot,

$$H_1(\widetilde{X}, \Lambda) \times H_1(\widetilde{X}, \Lambda) = \bigoplus_{\substack{k,\xi,\epsilon \\ \xi in S^1}} (\Lambda/_{p_\xi^k}, \epsilon)^{n_k,\xi,\epsilon} \oplus \bigoplus_{k,\eta} (\Lambda/_{p_\xi^k})^{m_k}$$

$$\textit{Let } \delta_\sigma(\xi) = \lim_{\varepsilon \to 0^+} \sigma(e^{2\pi i\varepsilon}\xi) - \sigma(e^{-2\pi i\varepsilon}\xi),$$

$$\textit{then } \sigma_j(\xi) = \sigma(\xi) - \frac{1}{2} \lim_{\varepsilon \to 0} \sigma(e^{2\pi i\varepsilon}\xi) + \sigma(e^{-2\pi i\varepsilon}\xi)$$

The jump at ξ is equal to $2 \sum\limits_{k_i \ odd} \epsilon_i$. The peak of the signature function is equal to $\sum\limits_{k_i \, even} \epsilon_i$.

$\square$

Lecture 9 — May 27, 2019

....

Definition 9.1
A square hermitian matrix A of size n.

field of fractions

Lecture 10 — June 3, 2019

Theorem 10.1
Let K be a knot and $u(K)$ its unknotting number. Let g_4 be a minimal four genus of a smooth surface S in B^4 such that $\partial S = K$. Then:

$$u(K) \geq g_4(K)$$

Proof. Recall that if $u(K) = u$ then K bounds a disk Δ with u ordinary double points.
Remove from Δ the two self intersecting and glue the Seifert surface for the Hopf link. The reality surface S has Euler characteristic $\chi(S) = 1 - 2u$. Therefore $g_4(S) = u$. $\square$

???????????????????

Example 10.1
The knot 8_{20} *is slice:* $\sigma \equiv 0$ *almost everywhere but* $\sigma(e^{\frac{2\pi i}{6}}) = +1$.

Surgery

Recall that $H_1(S^1 \times S^1, \mathbb{Z}) = \mathbb{Z}^3$. *As generators for* H_1 *we can set* $\alpha = [S^1 \times \{\mathrm{pt}\}]$ *and* $\beta = [\{\mathrm{pt}\} \times S^1]$. *Suppose* $\phi : S^1 \times S^1 \longrightarrow S^1 \times S^1$ *is a diffeomorphism. Consider an induced map on homology group:*

$$\begin{aligned} H_1(S^1 \times S^1, \mathbb{Z}) \ni \phi_*(\alpha) &= p\alpha + q\beta, \quad p, q \in \mathbb{Z}, \\ \phi_*(\beta) &= r\alpha + s\beta, \quad r, s \in \mathbb{Z}, \\ \phi_* &= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \end{aligned}$$

As ϕ_* *is diffeomorphis, it must be invertible over* $\mathbb{Z}$. *Then for a direction preserving diffeomorphism we have* $\det \phi_* = 1$. *Therefore* $\phi_* \in \mathrm{SL}(2, \mathbb{Z})$.

Theorem 10.2
Every such a matrix can be realized as a torus.

Proof. (I) Geometric reason

$$\begin{aligned} \phi_t : S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ S^1 \times \{\text{pt}\} &\longrightarrow \{\text{pt}\} \times S^1 \\ \{\text{pt}\} \times S^1 &\longrightarrow S^1 \times \{\text{pt}\} \\ (x, y) &\mapsto (-y, x) \end{aligned}$$

(II)

□

Lecture 11 balagan

Lecture 12 May 6, 2019

Definition 12.1
Let X be a knot complement. Then $H_1(X, \mathbb{Z}) \cong \mathbb{Z}$ and there exists an epimorphism $\pi_1(X) \overset{\phi}{\twoheadrightarrow} \mathbb{Z}$.
The infinite cyclic cover of a knot complement X is the cover associated with the epimorphism ϕ.

$$\widetilde{X} \longrightarrow\!\!\!\!\!\rightarrow X$$

Formal sums $\sum \phi_i(t)a_i + \sum \phi_j(t)\alpha_j$
finitely generated as a $\mathbb{Z}[t, t^{-1}]$ module.

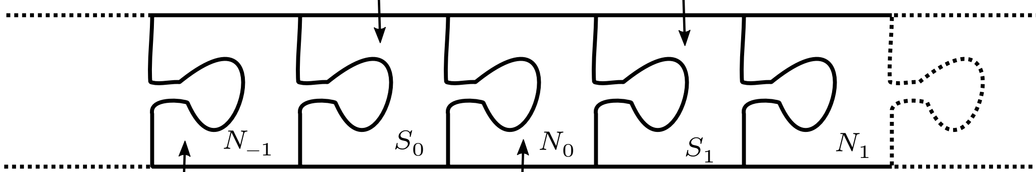


Figure 10: Infinite cyclic cover of a knot complement.

Let $v_{ij} = \operatorname{lk}(a_i, a_j^+)$. Then $V = \{v_ij\}_{i,j=1}^n$ is the Seifert matrix associated to the surface Σ and the basis $a_1, \dots, a_n$. Therefore $a_k^+ = \sum_j v_{jk}\alpha_j$. Then $\operatorname{lk}(a_i, a_k^+) = \operatorname{lk}(a_k^+, a_i) = \sum_j v_{jk} \operatorname{lk}(\alpha_j, a_i) = v_{ik}$. We also notice that $\operatorname{lk}(a_i, a_j^-) = \operatorname{lk}(a_i^+, a_j) = v_{ij}$ and $a_j^- = \sum_k v_{kj} t^{-1}\alpha_j$.
The homology of $\widetilde{X}$ is generated by $a_1, \dots, a_n$ and relations.

Definition 12.2
The Nakanishi index of a knot is the minimal number of generators of $H_1(\widetilde{X})$.

Remark about notation: sometimes one writes $H_1(X; \mathbb{Z}[t, t^{-1}])$ (what is also notation for twisted homology) instead of $H_1(\widetilde{X})$.
????????????????????
$\Sigma_?(K) \to S^3$?????
$H_1(\Sigma_?(K), \mathbb{Z}) = h$
$H \times H \longrightarrow {}^{\mathbb{Q}}/_{\mathbb{Z}}$
...

Let now $H = H_1(\widetilde{X})$. Can we define a paring?
Let $c, d \in H(\widetilde{X})$ (see Figure 12), Δ an Alexander polynomial. We know that $\Delta c = 0 \in H_1(\widetilde{X})$ (Alexander polynomial annihilates all possible elements). Let consider a surface F such that $\partial F = c$. Now consider intersection points $F \cdot d$. This points can exist in any N_k or S_k.

$$\frac{1}{\Delta} \sum_{j \in \mathbb{Z} t^{-j}} (F \cdot t^j d) \in {}^{\mathbb{Q}[t, t^{-1}]}/_{\mathbb{Z}[t, t^{-1}]}$$

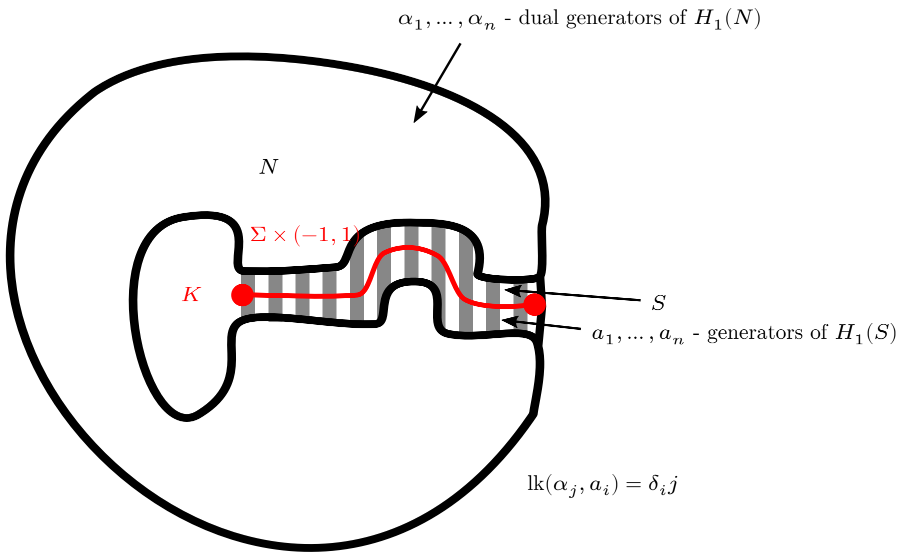


Figure 11: A knot complement.

?????????????

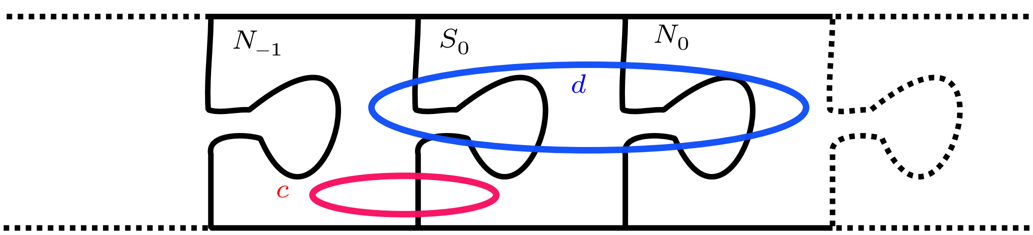


Figure 12: $c, d \in H_1(\widetilde{X})$.

Definition 12.3
The $\mathbb{Z}[t, t^{-1}]$ module $H_1(\widetilde{X})$ is called the Alexander module of knot K.

Let R be a PID, M a finitely generated R module. Let us consider

$$R^k \xrightarrow{A} R^n \twoheadrightarrow M,$$

where A is a $k \times n$ matrix, assume $k \geq n$. The order of M is the gcd of all determinants of the $n \times n$ minors of A. If $k = n$ then $\mathrm{ord}\, M = \det A$.

Theorem 12.1
Order of M doesn't depend on A.

For knots the order of the Alexander module is the Alexander polynomial.

Theorem 12.2

$$\forall x \in M : (\operatorname{ord} M)x = 0.$$

M is well defined up to a unit in R.

Blanchfield pairing

Lecture 13 balagan

Theorem 13.1
Let H_p be a p - torsion part of H. There exists an orthogonal decomposition of H_p:

$$H_p = H_{p,1} \oplus \cdots \oplus H_{p,r_p}.$$

$H_{p,i}$ is a cyclic module:

$$H_{p,i} = {}^{\mathbb{Z}[t,t^{-1}]}\big/_{p^{k_i}\mathbb{Z}[t,t^{-1}]}$$

The proof is the same as over $\mathbb{Z}$.