4.1.3 Minimum-variance unbiased estimator

Definition. Let A be a non-empty set of unbiased estimators of the parametric function $g(\theta)$, having finite variance (for any $\theta \in \Theta$). The $\widehat{g}_* \in A$ statistic is called a **minimum-variance unbiased estimator** (MVUE) of the parametric function $g(\theta)$, when

$$\forall \widehat{g} \in A \ \forall \theta \in \Theta : Var_\theta(\widehat{g}_*) \leqslant Var_\theta(\widehat{g}).$$

Theorem. If there is a minimum-variance unbiased estimator for the parametric function $g(\theta)$, it is determined uniquely (with accuracy to the set of measure zero).

4.1.4 Examples of estimators under different distributions

Let

$$\begin{aligned}
\bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i, \\
S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \\
\widetilde{S}^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.
\end{aligned}$$

- binomial distribution $b(m, p)$, $m \in \mathbb{N}$, $p \in (0,1)$

$$MLE(p) = UE(p) = MVUE(p) = \frac{1}{m}\bar{X}$$

- Poisson distribution $\pi(\lambda)$, $\lambda > 0$

$$MLE(\lambda) = UE(\lambda) = MVUE(\lambda) = \bar{X}$$

- uniform distribution $U(0, \theta)$, $\theta > 0$

$$\begin{aligned}
MLE(\theta) &= \max\{X_1, \dots, X_n\} \\
UE(\theta) = MVUE(\theta) &= \frac{n+1}{n}\max\{X_1, \dots, X_n\}
\end{aligned}$$

- normal distribution $N(\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$

$$\begin{aligned}
MLE(\mu) &= UE(\mu) = MVUE(\mu) = \bar{X} \\
MLE(\sigma^2) &= \widetilde{S}^2 \\
UE(\sigma^2) &= MVUE(\sigma^2) = S^2 \\
MLE(\sigma) &= \widetilde{S} \\
UE(\sigma) &= MVUE(\sigma) = \sqrt{\frac{n-1}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}S
\end{aligned}$$

- exponential distribution $Ex(\lambda)$, $\lambda > 0$

$$\begin{aligned}
MLE(\lambda) &= \frac{1}{\bar{X}} \\
UE(\lambda) = MVUE(\lambda) &= \frac{n-1}{n\bar{X}}
\end{aligned}$$